# Rules for integrands of the form $(a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n$

**1:** $\int (a + b \operatorname{Sin}[e + f x]) (c + d \operatorname{Sin}[e + f x]) \, dx$ when $b c - a d \neq 0$

- Derivation: Algebraic expansion

  Basis: $(a + b z) (c + d z) == \frac{1}{2} (2 a c + b d) + (b c + a d) z - \frac{1}{2} b d \left(1 - 2 z^2\right)$

- Rule: If $b c - a d \neq 0$, then

$$\int (a + b \operatorname{Sin}[e + f x]) (c + d \operatorname{Sin}[e + f x]) \, dx \longrightarrow \frac{(2 a c + b d) x}{2} - \frac{(b c + a d) \operatorname{Cos}[e + f x]}{f} - \frac{b d \operatorname{Cos}[e + f x] \operatorname{Sin}[e + f x]}{2 f}$$

- Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (2*a*c+b*d)*x/2 - (b*c+a*d)*Cos[e+f*x]/f - b*d*Cos[e+f*x]*Sin[e+f*x]/(2*f) /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

**2:** $\int \frac{a+b\,\text{Sin}[e+f\,x]}{c+d\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0$

Reference: G&R 2.551.2

Derivation: Algebraic expansion

Basis: $\frac{a+b\,z}{c+d\,z}==\frac{b}{d}-\frac{b\,c-a\,d}{d\,(c+d\,z)}$

Rule: If $b\,c-a\,d\neq 0$, then

$$\int \frac{a+b\,\text{Sin}[e+f\,x]}{c+d\,\text{Sin}[e+f\,x]}\,dx\;\longrightarrow\;\frac{b\,x}{d}-\frac{b\,c-a\,d}{d}\int \frac{1}{c+d\,\text{Sin}[e+f\,x]}\,dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  b*x/d - (b*c-a*d)/d*Int[1/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

**3.** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c + a d == 0 \land a^2 - b^2 == 0$

**1:** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c + a d == 0 \land a^2 - b^2 == 0 \land m \in \mathbb{Z}$

Derivation: Algebraic simplification

Basis: If $b c + a d == 0 \land a^2 - b^2 == 0$, then $(a + b \operatorname{Sin}[z]) (c + d \operatorname{Sin}[z]) == a c \operatorname{Cos}[z]^2$

Rule: If $b c + a d == 0 \land a^2 - b^2 == 0 \land m \in \mathbb{Z}$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx \longrightarrow a^m c^m \int \operatorname{Cos}[e + f x]^{2 m} (c + d \operatorname{Sin}[e + f x])^{n-m} \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^m*c^m*Int[Cos[e+f*x]^(2*m)*(c+d*Sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IntegerQ[m] && Not[IntegerQ[n] && (LtQ[m,0] && GtQ[n,0] || LtQ[0,n,m] || LtQ[m
```

2. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\land\,a^2-b^2==0\,\land\,m\notin\mathbb{Z}\,\land\,n\notin\mathbb{Z}$

1. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\land\,a^2-b^2==0\,\land\,m+\frac{1}{2}\in\mathbb{Z}^+$

1. $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\land\,a^2-b^2==0$

1: $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c+a\,d==0\,\land\,a^2-b^2==0$

Derivation: Piecewise constant extraction

Basis: If $b\,c+a\,d==0\,\land\,a^2-b^2==0$, then $\partial_x \frac{\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}==0$

Rule: If $b\,c+a\,d==0\,\land\,a^2-b^2==0$, then

$$\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \to \frac{a\,c\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\int \frac{\text{Cos}[e+f\,x]}{c+d\,\text{Sin}[e+f\,x]}\,dx$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  a*c*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])*Int[Cos[e+f*x]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0]
```

2: $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\land\,a^2-b^2==0\,\land\,n\neq-\frac{1}{2}$

Derivation: Doubly degenerate sine recurrence 1a with $p\to 0$

Rule: If $b\,c+a\,d==0\,\land\,a^2-b^2==0\,\land\,n\neq-\frac{1}{2}$, then

$$\int \sqrt{a + b \operatorname{Sin}[e + f x]}\ (c + d \operatorname{Sin}[e + f x])^n \, dx \to -\frac{2 b \operatorname{Cos}[e + f x]\ (c + d \operatorname{Sin}[e + f x])^n}{f\ (2 n + 1)\ \sqrt{a + b \operatorname{Sin}[e + f x]}}$$

## Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(f*(2*n+1)*Sqrt[a+b*Sin[e+f*x]]) /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && NeQ[n,-1/2]
```

2. $\int (a + b \operatorname{Sin}[e + f x])^m\ (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c + a d == 0\ \land\ a^2 - b^2 == 0\ \land\ m - \frac{1}{2} \in \mathbb{Z}^+$

   1: $\int (a + b \operatorname{Sin}[e + f x])^m\ (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c + a d == 0\ \land\ a^2 - b^2 == 0\ \land\ m - \frac{1}{2} \in \mathbb{Z}^+\ \land\ n < -1$

## Derivation: Doubly degenerate sine recurrence 1a with $p \to 0$

## Rule: If $b c + a d == 0\ \land\ a^2 - b^2 == 0\ \land\ m - \frac{1}{2} \in \mathbb{Z}^+\ \land\ n < -1$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m\ (c + d \operatorname{Sin}[e + f x])^n \, dx \to$$
$$-\frac{2 b \operatorname{Cos}[e + f x]\ (a + b \operatorname{Sin}[e + f x])^{m-1}\ (c + d \operatorname{Sin}[e + f x])^n}{f\ (2 n + 1)} - \frac{b\ (2 m - 1)}{d\ (2 n + 1)} \int (a + b \operatorname{Sin}[e + f x])^{m-1}\ (c + d \operatorname{Sin}[e + f x])^{n+1} \, dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(2*n+1)) -
  b*(2*m-1)/(d*(2*n+1))*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[m-1/2,0] && LtQ[n,-1] && Not[ILtQ[m+n,0]] && GtQ[2*m+n+1,0]]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\wedge\,a^2-b^2==0\,\wedge\,m-\frac{1}{2}\in\mathbb{Z}^+\,\wedge\,n\nless-1$

Derivation: Doubly degenerate sine recurrence 1b with $p\to 0$

Rule: If $b\,c+a\,d==0\,\wedge\,a^2-b^2==0\,\wedge\,m-\frac{1}{2}\in\mathbb{Z}^+\,\wedge\,n\nless-1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{b\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^{m-1}\,(c+d\,\mathrm{Sin}[e+f\,x])^n}{f\,(m+n)}+\frac{a\,(2\,m-1)}{m+n}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m-1}\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(m+n)) +
  a*(2*m-1)/(m+n)*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && IGtQ[m-1/2,0] && Not[LtQ[n,-1]] &&
  Not[IGtQ[n-1/2,0] && LtQ[n,m]] && Not[ILtQ[m+n,0] && GtQ[2*m+n+1,0]]
```

**2.** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\wedge\,a^2-b^2==0\,\wedge\,m+n\in\mathbb{Z}^-$

**1.** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d==0\,\wedge\,a^2-b^2==0\,\wedge\,m+n+1==0$

**1:** $\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c+a\,d==0\,\wedge\,a^2-b^2==0$

Derivation: Piecewise constant extraction

Basis: If $b\,c+a\,d==0\,\wedge\,a^2-b^2==0$, then $\partial_x\frac{\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}==0$

Rule: If $b\,c+a\,d==0\,\wedge\,a^2-b^2==0$, then

$$\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx \rightarrow \frac{\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\int \frac{1}{\mathrm{Cos}[e+f\,x]}\,dx$$

## Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])*Int[1/Cos[e+f*x],x] /;
FreeQ[{a,b,c,d,e,f},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d=0 \wedge a^2-b^2=0 \wedge m+n+1=0 \wedge m\neq -\frac{1}{2}$

## Derivation: Doubly degenerate sine recurrence 1c with $n \to -m-1,\ p \to 0$

## Rule: If $b\,c+a\,d=0 \wedge a^2-b^2=0 \wedge m+n+1=0 \wedge m\neq -\frac{1}{2}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx \rightarrow \frac{b\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n}{a\,f\,(2\,m+1)}$$

–

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && EqQ[m+n+1,0] && NeQ[m,-1/2]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c+a\,d=0 \wedge a^2-b^2=0 \wedge m+n+1\in\mathbb{Z}^- \wedge m\neq -\frac{1}{2}$

## Derivation: Doubly degenerate sine recurrence 1c with $p \to 0$

–

## Rule: If $b\,c+a\,d=0 \wedge a^2-b^2=0 \wedge m+n+1\in\mathbb{Z}^- \wedge m\neq -\frac{1}{2}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx \rightarrow$$

$$\frac{b \operatorname{Cos}[e+f x] \,(a+b \operatorname{Sin}[e+f x])^m \,(c+d \operatorname{Sin}[e+f x])^n}{a f \,(2 m+1)}+\frac{m+n+1}{a \,(2 m+1)} \int (a+b \operatorname{Sin}[e+f x])^{m+1} \,(c+d \operatorname{Sin}[e+f x])^n \, dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) +
  (m+n+1)/(a*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && ILtQ[Simplify[m+n+1],0] && NeQ[m,-1/2] &&
  (SumSimplerQ[m,1] || Not[SumSimplerQ[n,1]])
```

**3:** $\int (a+b \operatorname{Sin}[e+f x])^m \,(c+d \operatorname{Sin}[e+f x])^n \, dx$ when $b c+a d == 0 \land a^2-b^2 == 0 \land m<-1$

## Derivation: Doubly degenerate sine recurrence 1c with $p \to 0$

## Rule: If $b c+a d == 0 \land a^2-b^2 == 0 \land m<-1$, then

$$\int (a+b \operatorname{Sin}[e+f x])^m \,(c+d \operatorname{Sin}[e+f x])^n \, dx \longrightarrow$$

$$\frac{b \operatorname{Cos}[e+f x] \,(a+b \operatorname{Sin}[e+f x])^m \,(c+d \operatorname{Sin}[e+f x])^n}{a f \,(2 m+1)}+\frac{m+n+1}{a \,(2 m+1)} \int (a+b \operatorname{Sin}[e+f x])^{m+1} \,(c+d \operatorname{Sin}[e+f x])^n \, dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) +
  (m+n+1)/(a*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1] && Not[LtQ[m,n,-1]] && IntegersQ[2*m,2*n]
```

**4:** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c + a d == 0 \land a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land n \notin \mathbb{Z}$

## Derivation: Piecewise constant extraction

Basis: If $b c + a d == 0 \land a^2 - b^2 == 0$, then $\partial_x \frac{(a+b \operatorname{Sin}[e+f x])^m (c+d \operatorname{Sin}[e+f x])^m}{\operatorname{Cos}[e+f x]^{2 m}} == 0$

Rule: If $b c + a d == 0 \land a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land n \notin \mathbb{Z}$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx \longrightarrow$$
$$\frac{a^{\operatorname{IntPart}[m]} c^{\operatorname{IntPart}[m]} (a + b \operatorname{Sin}[e + f x])^{\operatorname{FracPart}[m]} (c + d \operatorname{Sin}[e + f x])^{\operatorname{FracPart}[m]}}{\operatorname{Cos}[e + f x]^{2 \operatorname{FracPart}[m]}} \int \operatorname{Cos}[e + f x]^{2 m} (c + d \operatorname{Sin}[e + f x])^{n-m} \, dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^IntPart[m]*c^IntPart[m]*(a+b*Sin[e+f*x])^FracPart[m]*(c+d*Sin[e+f*x])^FracPart[m]/Cos[e+f*x]^(2*FracPart[m])*
    Int[Cos[e+f*x]^(2*m)*(c+d*Sin[e+f*x])^(n-m),x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && EqQ[b*c+a*d,0] && EqQ[a^2-b^2,0] && (FractionQ[m] || Not[FractionQ[n]])
```

**4:** $\int \frac{(a + b \operatorname{Sin}[e + f x])^2}{c + d \operatorname{Sin}[e + f x]} \, dx$ when $b c - a d \neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{(a+b z)^2}{c+d z} == \frac{b^2 z}{d} + \frac{a^2 d-b (b c-2 a d) z}{d (c+d z)}$

## Rule: If $b c - a d \neq 0$, then

$$\int \frac{(a + b \operatorname{Sin}[e + f x])^2}{c + d \operatorname{Sin}[e + f x]} \, dx \longrightarrow -\frac{b^2 \operatorname{Cos}[e + f x]}{d f} + \frac{1}{d} \int \frac{a^2 d - b (b c - 2 a d) \operatorname{Sin}[e + f x]}{c + d \operatorname{Sin}[e + f x]} \, dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^2/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b^2*Cos[e+f*x]/(d*f) + 1/d*Int[Simp[a^2*d-b*(b*c-2*a*d)*Sin[e+f*x],x]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

**5:** $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,(c+d\,\text{Sin}[e+f\,x])}\,dx$ when $b\,c-a\,d\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{1}{(a+b\,z)\,(c+d\,z)} == \frac{b}{(b\,c-a\,d)\,(a+b\,z)} - \frac{d}{(b\,c-a\,d)\,(c+d\,z)}$

## Rule: If $b\,c-a\,d\neq 0$, then

$$\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,(c+d\,\text{Sin}[e+f\,x])}\,dx \longrightarrow \frac{b}{b\,c-a\,d}\int \frac{1}{a+b\,\text{Sin}[e+f\,x]}\,dx - \frac{d}{b\,c-a\,d}\int \frac{1}{c+d\,\text{Sin}[e+f\,x]}\,dx$$

## Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*(c_.+d_.*sin[e_.+f_.*x_])),x_Symbol] :=
  b/(b*c-a*d)*Int[1/(a+b*Sin[e+f*x]),x] - d/(b*c-a*d)*Int[1/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0]
```

6. $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x]) \, dx$ when $b c - a d \neq 0$

1: $\int (b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x]) \, dx$

- Derivation: Algebraic expansion

- Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0$, then

$$\int (b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x]) \, dx \longrightarrow c \int (b \operatorname{Sin}[e + f x])^m \, dx + \frac{d}{b} \int (b \operatorname{Sin}[e + f x])^{m+1} \, dx$$

- Program code:

```
Int[(b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  c*Int[(b*Sin[e+f*x])^m,x] + d/b*Int[(b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{b,c,d,e,f,m},x]
```

2. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0$

**1:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,a\,d\,m+b\,c\,(m+1)==0$

Derivation: Singly degenerate sine recurrence 2a with $A\to-\frac{a\,d\,m}{b\,(m+1)},\ B\to d,\ n\to 0,\ p\to 0$

Derivation: Singly degenerate sine recurrence 2c with $A\to-\frac{a\,d\,m}{b\,(m+1)},\ B\to d,\ n\to 0,\ p\to 0$

Note: If $a^2-b^2==0\,\land\,a\,d\,m+b\,c\,(m+1)==0$, then $m+1\neq 0$.

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,a\,d\,m+b\,c\,(m+1)==0$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])\,dx\longrightarrow-\frac{d\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^m}{f\,(m+1)}$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && EqQ[a*d*m+b*c*(m+1),0]
```

**2:** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])\, dx$ when $b c - a d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$

Derivation: Singly degenerate sine recurrence 2a with $A \to c,\ B \to d,\ n \to 0,\ p \to 0$

– Rule: If $b c - a d \neq 0 \wedge a^2 - b^2 == 0 \wedge m < -\frac{1}{2}$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])\, dx \longrightarrow$$
$$\frac{(b c - a d) \operatorname{Cos}[e + f x] (a + b \operatorname{Sin}[e + f x])^m}{a f (2 m + 1)} + \frac{a d m + b c (m + 1)}{a b (2 m + 1)} \int (a + b \operatorname{Sin}[e + f x])^{m+1}\, dx$$

– Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(a*f*(2*m+1)) +
  (a*d*m+b*c*(m+1))/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1/2]
```

**3:** $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x]) \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 == 0 \land m \nless -\frac{1}{2}$

- Derivation: Singly degenerate sine recurrence 2c with $A \to c$, $B \to d$, $n \to 0$, $p \to 0$

- Rule: If $b c - a d \neq 0 \land a^2 - b^2 == 0 \land m \nless -\frac{1}{2}$, then

$$\int (a + b \sin[e + f x])^m (c + d \sin[e + f x]) \, dx \to$$
$$-\frac{d \cos[e + f x] (a + b \sin[e + f x])^m}{f (m + 1)} + \frac{a d m + b c (m + 1)}{b (m + 1)} \int (a + b \sin[e + f x])^m \, dx$$

- Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) +
  (a*d*m+b*c*(m+1))/(b*(m+1))*Int[(a+b*Sin[e+f*x])^m,x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1/2]]
```

3. $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x]) \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0$

1. $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x]) \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land 2 m \in \mathbb{Z}$

**1:** $\int \frac{c + d \sin[e + f x]}{\sqrt{a + b \sin[e + f x]}} \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0$

- Derivation: Algebraic expansion

Basis: $c + d z == \frac{b c - a d}{b} + \frac{d}{b} (a + b z)$

- Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0$, then

$$\int \frac{c + d \operatorname{Sin}[e + f x]}{\sqrt{a + b \operatorname{Sin}[e + f x]}} \, dx \longrightarrow \frac{b c - a d}{b} \int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f x]}} \, dx + \frac{d}{b} \int \sqrt{a + b \operatorname{Sin}[e + f x]} \, dx$$

## Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  (b*c-a*d)/b*Int[1/Sqrt[a+b*Sin[e+f*x]],x] + d/b*Int[Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

**2:** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x]) \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land m > 0 \land 2 m \in \mathbb{Z}$

## Reference: G&R 2.551.1 inverted

## Derivation: Nondegenerate sine recurrence 1b with $A \to a c$, $B \to b c + a d$, $C \to b d$, $m \to 0$, $n \to n - 1$, $p \to 0$

## Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land m > 0 \land 2 m \in \mathbb{Z}$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x]) \, dx \longrightarrow$$

$$-\frac{d \operatorname{Cos}[e + f x] (a + b \operatorname{Sin}[e + f x])^m}{f (m + 1)} + \frac{1}{m + 1} \int (a + b \operatorname{Sin}[e + f x])^{m-1} (b d m + a c (m + 1) + (a d m + b c (m + 1)) \operatorname{Sin}[e + f x]) \, dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(f*(m+1)) +
  1/(m+1)*Int[(a+b*Sin[e+f*x])^(m-1)*Simp[b*d*m+a*c*(m+1)+(a*d*m+b*c*(m+1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && GtQ[m,0] && IntegerQ[2*m]
```

**3:** $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx$ when $b\,c - a\,d \neq 0\,\land\,a^2 - b^2 \neq 0\,\land\,m < -1\,\land\,2\,m \in \mathbb{Z}$

Reference: G&R 2.551.1

Derivation: Nondegenerate sine recurrence 1a with $A \to c,\ B \to d,\ C \to 0,\ n \to 0,\ p \to 0$

Rule: If $b\,c - a\,d \neq 0\,\land\,a^2 - b^2 \neq 0\,\land\,m < -1\,\land\,2\,m \in \mathbb{Z}$, then

$$\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx \longrightarrow$$
$$-\frac{(b\,c - a\,d)\,\text{Cos}[e + f\,x]\,(a + b\,\text{Sin}[e + f\,x])^{m+1}}{f\,(m + 1)\,(a^2 - b^2)} + \frac{1}{(m + 1)\,(a^2 - b^2)} \int (a + b\,\text{Sin}[e + f\,x])^{m+1}\,((a\,c - b\,d)\,(m + 1) - (b\,c - a\,d)\,(m + 2)\,\text{Sin}[e + f\,x])\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*Simp[(a*c-b*d)*(m+1)-(b*c-a*d)*(m+2)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && LtQ[m,-1] && IntegerQ[2*m]
```

**2.** $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx$ when $b\,c - a\,d \neq 0\,\land\,a^2 - b^2 \neq 0\,\land\,2\,m \notin \mathbb{Z}$

**1:** $\int (a + b\,\text{Sin}[e + f\,x])^m\,(c + d\,\text{Sin}[e + f\,x])\,dx$ when $b\,c - a\,d \neq 0\,\land\,a^2 - b^2 \neq 0\,\land\,2\,m \notin \mathbb{Z}\,\land\,c^2 - d^2 == 0$

Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{\text{Cos}[e+f\,x]}{\sqrt{1+\text{Sin}[e+f\,x]}\,\sqrt{1-\text{Sin}[e+f\,x]}} == 0$

Basis: $\text{Cos}[e + f\,x] == \frac{1}{f}\,\partial_x\,\text{Sin}[e + f\,x]$

Rule: If $b\,c - a\,d \neq 0\,\land\,a^2 - b^2 \neq 0\,\land\,2\,m \notin \mathbb{Z}\,\land\,c^2 - d^2 == 0$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])\,dx\,\longrightarrow$$

$$\frac{c\,\mathrm{Cos}[e+f\,x]}{\sqrt{1+\mathrm{Sin}[e+f\,x]}\,\sqrt{1-\mathrm{Sin}[e+f\,x]}}\int\frac{\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,\sqrt{1+\frac{d}{c}\,\mathrm{Sin}[e+f\,x]}}{\sqrt{1-\frac{d}{c}\,\mathrm{Sin}[e+f\,x]}}\,dx\,\longrightarrow$$

$$\frac{c\,\mathrm{Cos}[e+f\,x]}{f\,\sqrt{1+\mathrm{Sin}[e+f\,x]}\,\sqrt{1-\mathrm{Sin}[e+f\,x]}}\,\mathrm{Subst}\Big[\int\frac{(a+b\,x)^m\,\sqrt{1+\frac{d}{c}\,x}}{\sqrt{1-\frac{d}{c}\,x}}\,dx,\,x,\,\mathrm{Sin}[e+f\,x]\Big]$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  c*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*Subst[Int[(a+b*x)^m*Sqrt[1+d/c*x]/Sqrt[1-d/c*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && Not[IntegerQ[2*m]] && EqQ[c^2-d^2,0]
```

**2:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,2\,m\notin\mathbb{Z}\,\wedge\,c^2-d^2\neq 0$

## Derivation: Algebraic expansion

Basis: $c+d\,z==\frac{b\,c-a\,d}{b}+\frac{d}{b}\,(a+b\,z)$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])\,dx\,\longrightarrow\,\frac{b\,c-a\,d}{b}\int (a+b\,\text{Sin}[e+f\,x])^m\,dx+\frac{d}{b}\int (a+b\,\text{Sin}[e+f\,x])^{m+1}\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  (b*c-a*d)/b*Int[(a+b*Sin[e+f*x])^m,x] + d/b*Int[(a+b*Sin[e+f*x])^(m+1),x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0]
```

7. $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0$

**1:** $\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx$ when $a^2 - b^2 == 0 \land m \in \mathbb{Z}^+$

Derivation: Algebraic expansion

Rule: If $a^2 - b^2 == 0 \land m \in \mathbb{Z}^+$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx \longrightarrow \int \operatorname{ExpandTrig}\left[(a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n, x\right] dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && EqQ[a^2-b^2,0] && IGtQ[m,0] && RationalQ[n]
```

**2.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^2\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0$

**1.** $\int \text{Sin}[e+f\,x]^2\,(a+b\,\text{Sin}[e+f\,x])^m\,dx$ when $a^2-b^2==0$

**1:** $\int \text{Sin}[e+f\,x]^2\,(a+b\,\text{Sin}[e+f\,x])^m\,dx$ when $a^2-b^2==0\,\land\,m<-\frac{1}{2}$

## Derivation: ???

## Rule: If $a^2-b^2==0\,\land\,m<-\frac{1}{2}$, then

$$\int \text{Sin}[e+f\,x]^2\,(a+b\,\text{Sin}[e+f\,x])^m\,dx\,\longrightarrow$$
$$\frac{b\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^m}{a\,f\,(2\,m+1)}-\frac{1}{a^2\,(2\,m+1)}\int (a+b\,\text{Sin}[e+f\,x])^{m+1}\,(a\,m-b\,(2\,m+1)\,\text{Sin}[e+f\,x])\,dx$$

## Program code:

```
Int[sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m/(a*f*(2*m+1)) -
  1/(a^2*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(a*m-b*(2*m+1)*Sin[e+f*x]),x] /;
FreeQ[{a,b,e,f},x] && EqQ[a^2-b^2,0] && LtQ[m,-1/2]
```

**2:** $\int \text{Sin}[e+f\,x]^2\,(a+b\,\text{Sin}[e+f\,x])^m\,dx$ when $a^2-b^2=0\,\land\,m\nless-\frac{1}{2}$

Derivation: Nondegenerate sine recurrence 1b with $A\to a^2$, $B\to 2\,a\,b$, $C\to b^2$, $m\to 0$, $p\to 0$

Rule: If $a^2-b^2=0\,\land\,m\nless-\frac{1}{2}$, then

$$\int \text{Sin}[e+f\,x]^2\,(a+b\,\text{Sin}[e+f\,x])^m\,dx\,\longrightarrow$$
$$-\frac{\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m+1}}{b\,f\,(m+2)}+\frac{1}{b\,(m+2)}\int (a+b\,\text{Sin}[e+f\,x])^m\,(b\,(m+1)-a\,\text{Sin}[e+f\,x])\,dx$$

Program code:

```
Int[sin[e_.+f_.*x_]^2*(a_+b_.*sin[e_.+f_.*x_])^m_,x_Symbol] :=
  -Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*(b*(m+1)-a*Sin[e+f*x]),x] /;
FreeQ[{a,b,e,f,m},x] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1/2]]
```

2. $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0$

**1:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,m<-1$

Derivation: Singly degenerate sine recurrence 2a with $A\to c,\ B\to d,\ n\to 1,\ p\to 0$

– Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,m<-1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx\,\longrightarrow$$

$$\frac{(b\,c-a\,d)\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])}{a\,f\,(2\,m+1)}+$$

$$\frac{1}{a\,b\,(2\,m+1)}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m+1}\,\left(a\,c\,d\,(m-1)+b\,\left(d^2+c^2\,(m+1)\right)+d\,(a\,d\,(m-1)+b\,c\,(m+2))\,\mathrm{Sin}[e+f\,x]\right)dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])/(a*f*(2*m+1)) +
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*Simp[a*c*d*(m-1)+b*(d^2+c^2*(m+1))+d*(a*d*(m-1)+b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && LtQ[m,-1]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,m\nless -1$

Derivation: Nondegenerate sine recurrence 1b with $A\to a^2,\ B\to 2\,a\,b,\ C\to b^2,\ m\to 0,\ p\to 0$

– Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,m\nless -1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx\,\longrightarrow$$

$$-\frac{d^2\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^{m+1}}{b\,f\,(m+2)}+$$

$$\frac{1}{b\,(m+2)}\int(a+b\,\text{Sin}[e+f\,x])^m\,\left(b\,\left(d^2\,(m+1)+c^2\,(m+2)\right)-d\,(a\,d-2\,b\,c\,(m+2))\,\text{Sin}[e+f\,x]\right)\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  -d^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*Simp[b*(d^2*(m+1)+c^2*(m+2))-d*(a*d-2*b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && Not[LtQ[m,-1]]
```

3. $\int(a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0\,\land\,m>1$

1: $\int(a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0\,\land\,m>1\,\land\,n<-1$

## Derivation: Singly degenerate sine recurrence 1a with $A\to a,\ B\to b,\ m\to m-1,\ p\to0$

## Rule: If $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0\,\land\,m>1\,\land\,n<-1$, then

$$\int(a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$

$$-\frac{b^2\,(b\,c-a\,d)\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}}{d\,f\,(n+1)\,(b\,c+a\,d)}+$$

$$\frac{b^2}{d\,(n+1)\,(b\,c+a\,d)}\int(a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}\,(a\,c\,(m-2)-b\,d\,(m-2\,n-4)-(b\,c\,(m-1)-a\,d\,(m+2\,n+1))\,\text{Sin}[e+f\,x])\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(n+1)*(b*c+a*d)) +
  b^2/(d*(n+1)*(b*c+a*d))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)*
    Simp[a*c*(m-2)-b*d*(m-2*n-4)-(b*c*(m-1)-a*d*(m+2*n+1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,1] && LtQ[n,-1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m+1/2] || IntegerQ[m] && EqQ[c,0])
```

**2:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m>1\,\wedge\,n\nless -1$

Derivation: Singly degenerate sine recurrence 1b with $A\to a,\ B\to b,\ m\to m-1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m>1\,\wedge\,n\nless -1$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{b^2\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}}{d\,f\,(m+n)}+$$
$$\frac{1}{d\,(m+n)}\int (a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^n\,\cdot$$
$$(a\,b\,c\,(m-2)+b^2\,d\,(n+1)+a^2\,d\,(m+n)-b\,(b\,c\,(m-1)-a\,d\,(3\,m+2\,n-2))\,\text{Sin}[e+f\,x])\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^n*
    Simp[a*b*c*(m-2)+b^2*d*(n+1)+a^2*d*(m+n)-b*(b*c*(m-1)-a*d*(3*m+2*n-2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,1] && Not[LtQ[n,-1]] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m+1/2] || IntegerQ[m] && EqQ[c,0])
```

**4.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1$

**1.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n>0$

**1:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,0<n<1$

Derivation: Singly degenerate sine recurrence 2a with $A\to 1,\ B\to 0,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2=0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,0<n<1$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx \longrightarrow$$
$$\frac{b \operatorname{Cos}[e + f x] (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n}{a f (2 m + 1)} -$$
$$\frac{1}{a b (2 m + 1)} \int (a + b \operatorname{Sin}[e + f x])^{m+1} (c + d \operatorname{Sin}[e + f x])^{n-1} (a d n - b c (m + 1) - b d (m + n + 1) \operatorname{Sin}[e + f x]) \, dx$$

–

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n/(a*f*(2*m+1)) -
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)*Simp[a*d*n-b*c*(m+1)-b*d*(m+n+1)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[0,n,1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n>1$

Derivation: Singly degenerate sine recurrence 2a with $A\to c,\ B\to d,\ n\to n-1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n>1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$

$$\frac{(b\,c-a\,d)\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-1}}{a\,f\,(2\,m+1)}+$$

$$\frac{1}{a\,b\,(2\,m+1)}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m+1}\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-2}\,\left(b\,\left(c^2\,(m+1)+d^2\,(n-1)\right)+a\,c\,d\,(m-n+1)+d\,(a\,d\,(m-n+1)+b\,c\,(m+n))\,\mathrm{Sin}[e+f\,x]\right)dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-1)/(a*f*(2*m+1)) +
  1/(a*b*(2*m+1))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-2)*
    Simp[b*(c^2*(m+1)+d^2*(n-1))+a*c*d*(m-n+1)+d*(a*d*(m-n+1)+b*c*(m+n))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && GtQ[n,1] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n\not>0$

Derivation: Singly degenerate sine recurrence 2b with $A\to 1,\ B\to 0,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n\not>0$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$

$$\frac{b^2\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^{n+1}}{a\,f\,(2\,m+1)\,(b\,c-a\,d)}+$$

$$\frac{1}{a\,(2\,m+1)\,(b\,c-a\,d)}\int(a+b\,\text{Sin}[e+f\,x])^{m+1}\,(c+d\,\text{Sin}[e+f\,x])^n\,(b\,c\,(m+1)-a\,d\,(2\,m+n+2)+b\,d\,(m+n+2)\,\text{Sin}[e+f\,x])\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n+1)/(a*f*(2*m+1)*(b*c-a*d)) +
  1/(a*(2*m+1)*(b*c-a*d))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n*
    Simp[b*c*(m+1)-a*d*(2*m+n+2)+b*d*(m+n+2)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && Not[GtQ[n,0]] &&
  (IntegersQ[2*m,2*n] || IntegerQ[m] && EqQ[c,0])
```

5. $\int\frac{(c+d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0$

1: $\int\frac{(c+d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0\,\land\,n>1$

Derivation: Singly degenerate sine recurrence 2a with $A\to c,\ B\to d,\ m\to-1,\ n\to n-1,\ p\to0$

Rule: If $b\,c-a\,d\neq0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq0\,\land\,n>1$, then

$$\int\frac{(c+d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx\,\longrightarrow$$

$$-\frac{(b\,c-a\,d)\,\text{Cos}[e+f\,x]\,(c+d\,\text{Sin}[e+f\,x])^{n-1}}{a\,f\,(a+b\,\text{Sin}[e+f\,x])}-\frac{d}{a\,b}\int(c+d\,\text{Sin}[e+f\,x])^{n-2}\,(b\,d\,(n-1)-a\,c\,n+(b\,c\,(n-1)-a\,d\,n)\,\text{Sin}[e+f\,x])\,dx$$

## Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n-1)/(a*f*(a+b*Sin[e+f*x])) -
  d/(a*b)*Int[(c+d*Sin[e+f*x])^(n-2)*Simp[b*d*(n-1)-a*c*n+(b*c*(n-1)-a*d*n)*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && (IntegerQ[2*n] || EqQ[c,0])
```

**2:** $\int \frac{(c + d \operatorname{Sin}[e + f x])^n}{a + b \operatorname{Sin}[e + f x]} \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land n < 0$

Derivation: Singly degenerate sine recurrence 2b with $A \to 1,\ B \to 0,\ m \to -1,\ p \to 0$

Rule: If $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land n < 0$, then

$$\int \frac{(c + d \operatorname{Sin}[e + f x])^n}{a + b \operatorname{Sin}[e + f x]} \, dx \longrightarrow$$

$$-\frac{b^2 \operatorname{Cos}[e + f x] \, (c + d \operatorname{Sin}[e + f x])^{n+1}}{a f (b c - a d) \, (a + b \operatorname{Sin}[e + f x])} + \frac{d}{a (b c - a d)} \int (c + d \operatorname{Sin}[e + f x])^n \, (a n - b (n + 1) \operatorname{Sin}[e + f x]) \, dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b^2*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(a*f*(b*c-a*d)*(a+b*Sin[e+f*x])) +
  d/(a*(b*c-a*d))*Int[(c+d*Sin[e+f*x])^n*(a*n-b*(n+1)*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,0] && (IntegerQ[2*n] || EqQ[c,0])
```

**3:** $\int \frac{(c+d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0$

Derivation: Singly degenerate sine recurrence 2a with $A\to 1,\ B\to 0,\ m\to -1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0$, then

$$\int \frac{(c+d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx\,\longrightarrow$$

$$-\frac{b\,\text{Cos}[e+f\,x]\,(c+d\,\text{Sin}[e+f\,x])^n}{a\,f\,(a+b\,\text{Sin}[e+f\,x])}+\frac{d\,n}{a\,b}\int (c+d\,\text{Sin}[e+f\,x])^{n-1}\,(a-b\,\text{Sin}[e+f\,x])\,dx$$

Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(a*f*(a+b*Sin[e+f*x])) +
  d*n/(a*b)*Int[(c+d*Sin[e+f*x])^(n-1)*(a-b*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && (IntegerQ[2*n] || EqQ[c,0])
```

**6.** $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0$

**1.** $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0\wedge 2\,n\in\mathbb{Z}$

**1:** $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0\wedge n>0$

Derivation: Singly degenerate sine recurrence 1b with $A\to c,\ B\to d,\ m\to \frac{1}{2},\ n\to n-1,\ p\to 0$ and algebraic simplification

Rule: If $b\,c-a\,d\neq 0\wedge a^2-b^2==0\wedge c^2-d^2\neq 0\wedge n>0$, then

$$\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^n\,dx \longrightarrow$$
$$-\frac{2\,b\,\text{Cos}[e+f\,x]\,\left(c+d\,\text{Sin}[e+f\,x]\right)^n}{f\,(2\,n+1)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}}+\frac{2\,n\,(b\,c+a\,d)}{b\,(2\,n+1)}\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^{n-1}\,dx$$

## Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -2*b*Cos[e+f*x]*(c+d*Sin[e+f*x])^n/(f*(2*n+1)*Sqrt[a+b*Sin[e+f*x]]) +
  2*n*(b*c+a*d)/(b*(2*n+1))*Int[Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])^(n-1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,0] && IntegerQ[2*n]
```

2. $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^n\,dx$ when $b\,c-a\,d\neq 0\land a^2-b^2==0\land c^2-d^2\neq 0\land n<-1$

1: $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\left(c+d\,\text{Sin}[e+f\,x]\right)^{3/2}}\,dx$ when $b\,c-a\,d\neq 0\land a^2-b^2==0\land c^2-d^2\neq 0$

Derivation: Singly degenerate sine recurrence 1a with $A\to 1,\ B\to 0,\ m\to \frac{1}{2},\ n\to -\frac{3}{2}\ p\to 0$

Derivation: Singly degenerate sine recurrence 1c with $A\to a,\ B\to b,\ m\to -\frac{1}{2},\ n\to -\frac{3}{2},\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\land a^2-b^2==0\land c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\left(c+d\,\text{Sin}[e+f\,x]\right)^{3/2}}\,dx \longrightarrow -\frac{2\,b^2\,\text{Cos}[e+f\,x]}{f\,(b\,c+a\,d)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}$$

## Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/(c_.+d_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  -2*b^2*Cos[e+f*x]/(f*(b*c+a*d)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]) /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^n\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n<-1$

Derivation: Singly degenerate sine recurrence 1c with $A\to a,\ B\to b,\ m\to -\frac{1}{2},\ p\to 0$ and algebraic simplification

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n<-1$, then

$$\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^n\,dx\,\longrightarrow$$

$$\frac{(b\,c-a\,d)\,\text{Cos}[e+f\,x]\,\left(c+d\,\text{Sin}[e+f\,x]\right)^{n+1}}{f\,(n+1)\,\left(c^2-d^2\right)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}}+\frac{(2\,n+3)\,(b\,c-a\,d)}{2\,b\,(n+1)\,\left(c^2-d^2\right)}\int \sqrt{a+b\,\text{Sin}[e+f\,x]}\,\left(c+d\,\text{Sin}[e+f\,x]\right)^{n+1}\,dx$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  (b*c-a*d)*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(f*(n+1)*(c^2-d^2)*Sqrt[a+b*Sin[e+f*x]]) +
  (2*n+3)*(b*c-a*d)/(2*b*(n+1)*(c^2-d^2))*Int[Sqrt[a+b*Sin[e+f*x]]*(c+d*Sin[e+f*x])^(n+1),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,-1] && NeQ[2*n+3,0] && IntegerQ[2*n]
```

**3:** $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{c+d\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$

Author: Martin Welz on 24 June 2011; generalized by Albert Rich 14 April 2014

Derivation: Integration by substitution

Basis: If $a^2-b^2==0$, then $\frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{c+d\,\text{Sin}[e+f\,x]}==-\frac{2\,b}{f}\,\text{Subst}\left[\frac{1}{b\,c+a\,d-d\,x^2},\,x,\,\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\right]\,\partial_x\,\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{c+d\,\text{Sin}[e+f\,x]}\,dx \to -\frac{2\,b}{f}\,\text{Subst}\Big[\int \frac{1}{b\,c+a\,d-d\,x^2}\,dx,\,x,\,\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\Big]$$

– Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  -2*b/f*Subst[Int[1/(b*c+a*d-d*x^2),x],x,b*Cos[e+f*x]/Sqrt[a+b*Sin[e+f*x]]] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

4. $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2=0\,\land\,c^2-d^2\neq 0$

1: $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2=0\,\land\,d=\frac{a}{b}$

Author: Martin Welz on 24 June 2011

– Derivation: Integration by substitution

Basis: If $a^2-b^2=0\,\land\,d=\frac{a}{b}$, then $\frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}} = -\frac{2}{f}\,\text{Subst}\Big[\frac{1}{\sqrt{1-\frac{x^2}{a}}},\,x,\,\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\Big]\,\partial_x\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}$

– Rule: If $a^2-b^2=0\,\land\,d=\frac{a}{b}$, then

$$\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to -\frac{2}{f}\,\text{Subst}\Big[\int \frac{1}{\sqrt{1-\frac{x^2}{a}}}\,dx,\,x,\,\frac{b\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\Big]$$

– Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2/f*Subst[Int[1/Sqrt[1-x^2/a],x],x,b*Cos[e+f*x]/Sqrt[a+b*Sin[e+f*x]]] /;
FreeQ[{a,b,d,e,f},x] && EqQ[a^2-b^2,0] && EqQ[d,a/b]
```

**2:** $\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\ \land\ a^2-b^2==0\ \land\ c^2-d^2\neq 0$

Author: Martin Welz on 10 March 2011

Derivation: Integration by substitution

Basis: If $a^2-b^2==0\ \land\ c^2-d^2\neq 0$, then $\frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}==-\frac{2\,b}{f}\,\mathrm{Subst}\left[\frac{1}{b+d\,x^2},\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\right]\partial_x\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}$

Note: The above identity is not valid if $b\,c-a\,d\neq 0\ \land\ a^2-b^2==0\ \land\ c^2-d^2==0$, since the derivative vanishes!

Rule: If $b\,c-a\,d\neq 0\ \land\ a^2-b^2==0\ \land\ c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx \longrightarrow -\frac{2\,b}{f}\,\mathrm{Subst}\left[\int \frac{1}{b+d\,x^2}\,dx,\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\right]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2*b/f*Subst[Int[1/(b+d*x^2),x],x,b*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\left(c+d\,\mathrm{Sin}[e+f\,x]\right)^n\,dx$ when $b\,c-a\,d\neq0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq0\,\wedge\,2\,n\notin\mathbb{Z}$

## Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2-b^2==0$, then $\partial_x\frac{\mathrm{Cos}[e+f\,x]}{\sqrt{a-b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}==0$

Basis: $\mathrm{Cos}[e+f\,x]==\frac{1}{f}\,\partial_x\,\mathrm{Sin}[e+f\,x]$

Rule: If $b\,c-a\,d\neq0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq0\,\wedge\,2\,n\notin\mathbb{Z}$, then

$$\int\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\left(c+d\,\mathrm{Sin}[e+f\,x]\right)^n\,dx\,\rightarrow$$

$$\frac{a^2\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{a-b\,\mathrm{Sin}[e+f\,x]}}\int\frac{\mathrm{Cos}[e+f\,x]\,\left(c+d\,\mathrm{Sin}[e+f\,x]\right)^n}{\sqrt{a-b\,\mathrm{Sin}[e+f\,x]}}\,dx\,\rightarrow$$

$$\frac{a^2\,\mathrm{Cos}[e+f\,x]}{f\,\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{a-b\,\mathrm{Sin}[e+f\,x]}}\,\mathrm{Subst}\left[\int\frac{(c+d\,x)^n}{\sqrt{a-b\,x}}\,dx,\,x,\,\mathrm{Sin}[e+f\,x]\right]$$

## Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  a^2*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*Subst[Int[(c+d*x)^n/Sqrt[a-b*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[IntegerQ[2*n]]
```

**7.** $\int\frac{\left(c+d\,\mathrm{Sin}[e+f\,x]\right)^n}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq0$

**1.** $\int\frac{\left(c+d\,\mathrm{Sin}[e+f\,x]\right)^n}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq0\,\wedge\,n>0$

**1:** $\int \frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{\sqrt{c+d\,z}}{\sqrt{a+b\,z}}==\frac{b\,c-a\,d}{b\,\sqrt{a+b\,z}\,\sqrt{c+d\,z}}+\frac{d\,\sqrt{a+b\,z}}{b\,\sqrt{c+d\,z}}$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx\,\longrightarrow\,\frac{d}{b}\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx+\frac{b\,c-a\,d}{b}\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[Sqrt[c_.+d_.*sin[e_.+f_.*x_]]/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  d/b*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int \frac{(c+d\,\text{Sin}[e+f\,x])^n}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n>1$

## Derivation: Singly degenerate sine recurrence 2c with $A\to c,\ B\to d,\ m\to\frac{1}{2},\ n\to n-1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n>1$, then

$$\int \frac{(c+d\,\text{Sin}[e+f\,x])^n}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx\,\longrightarrow$$

$$-\frac{2\,d\,\text{Cos}[e+f\,x]\,(c+d\,\text{Sin}[e+f\,x])^{n-1}}{f\,(2\,n-1)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,-$$

$$\frac{1}{b\,(2\,n-1)}\int\frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,(c+d\,\text{Sin}[e+f\,x])^{n-2}\,\left(a\,c\,d-b\,\left(2\,d^2\,(n-1)+c^2\,(2\,n-1)\right)+d\,(a\,d-b\,c\,(4\,n-3))\,\text{Sin}[e+f\,x]\right)\,dx$$

## Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -2*d*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n-1)/(f*(2*n-1)*Sqrt[a+b*Sin[e+f*x]]) -
  1/(b*(2*n-1))*Int[(c+d*Sin[e+f*x])^(n-2)/Sqrt[a+b*Sin[e+f*x]]*
    Simp[a*c*d-b*(2*d^2*(n-1)+c^2*(2*n-1))+d*(a*d-b*c*(4*n-3))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && IntegerQ[2*n]
```

**2:** $\int\frac{(c+d\,\text{Sin}[e+f\,x])^n}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,n<-1$

## Derivation: Singly degenerate sine recurrence 1c with $A\to 1,\ B\to 0,\ p\to 0$

## Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,n<-1$, then

$$\int\frac{(c+d\,\text{Sin}[e+f\,x])^n}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx\,\longrightarrow$$

$$-\frac{d\,\text{Cos}[e+f\,x]\,(c+d\,\text{Sin}[e+f\,x])^{n+1}}{f\,(n+1)\,(c^2-d^2)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}}-\frac{1}{2\,b\,(n+1)\,(c^2-d^2)}\int\frac{(c+d\,\text{Sin}[e+f\,x])^{n+1}\,(a\,d-2\,b\,c\,(n+1)+b\,d\,(2\,n+3)\,\text{Sin}[e+f\,x])}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[(c_.+d_.*sin[e_.+f_.*x_])^n_/Sqrt[a_+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -d*Cos[e+f*x]*(c+d*Sin[e+f*x])^(n+1)/(f*(n+1)*(c^2-d^2)*Sqrt[a+b*Sin[e+f*x]]) -
  1/(2*b*(n+1)*(c^2-d^2))*Int[(c+d*Sin[e+f*x])^(n+1)*Simp[a*d-2*b*c*(n+1)+b*d*(2*n+3)*Sin[e+f*x],x]/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[n,-1] && IntegerQ[2*n]
```

3: $\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\left(c+d\,\mathrm{Sin}[e+f\,x]\right)}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0$

## Derivation: Algebraic expansion

- Basis: $\frac{1}{\sqrt{a+b\,z}\,(c+d\,z)}==\frac{b}{(b\,c-a\,d)\,\sqrt{a+b\,z}}-\frac{d\,\sqrt{a+b\,z}}{(b\,c-a\,d)\,(c+d\,z)}$

- Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0$, then

$$\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\left(c+d\,\mathrm{Sin}[e+f\,x]\right)}\,dx\to\frac{b}{b\,c-a\,d}\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\,dx-\frac{d}{b\,c-a\,d}\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{c+d\,\mathrm{Sin}[e+f\,x]}\,dx$$

- Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*(c_.+d_.*sin[e_.+f_.*x_])),x_Symbol] :=
  b/(b*c-a*d)*Int[1/Sqrt[a+b*Sin[e+f*x]],x] - d/(b*c-a*d)*Int[Sqrt[a+b*Sin[e+f*x]]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

4. $\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0$

1: $\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2==0\,\wedge\,d==\frac{a}{b}\,\wedge\,a>0$

Author: Martin Welz on 24 June 2011

Derivation: Integration by substitution

Basis: If $a^2-b^2==0\,\wedge\,d==\frac{a}{b}\,\wedge\,a>0$, then $\frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{d\,\mathrm{Sin}[e+f\,x]}}==-\frac{\sqrt{2}}{\sqrt{a}\,f}\,\mathrm{Subst}\left[\frac{1}{\sqrt{1-x^2}},\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{a+b\,\mathrm{Sin}[e+f\,x]}\right]\,\partial_x\frac{b\,\mathrm{Cos}[e+f\,x]}{a+b\,\mathrm{Sin}[e+f\,x]}$

Basis: $F(z\mid 0)==z$

Note: This is a special case of the rule for $a^2\neq b^2$.

Rule: If $a^2-b^2==0\,\wedge\,d==\frac{a}{b}\,\wedge\,a>0$, then

$$\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx\,\rightarrow\,-\frac{\sqrt{2}}{\sqrt{a}\,f}\,\mathrm{Subst}\left[\int\frac{1}{\sqrt{1-x^2}}\,dx,\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{a+b\,\mathrm{Sin}[e+f\,x]}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -Sqrt[2]/(Sqrt[a]*f)*Subst[Int[1/Sqrt[1-x^2],x],x,b*Cos[e+f*x]/(a+b*Sin[e+f*x])] /;
FreeQ[{a,b,d,e,f},x] && EqQ[a^2-b^2,0] && EqQ[d,a/b] && GtQ[a,0]
```

**2:** $\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$

Author: Martin Welz on 10 March 2011

Derivation: Integration by substitution

Basis: If $a^2-b^2==0$, then $\frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}==-\frac{2a}{f}\,\mathrm{Subst}\left[\frac{1}{2b^2-(a\,c-b\,d)\,x^2},\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\right]\partial_x\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}$

Note: The above identity is not valid if $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2==0$, since the derivative vanishes!

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0$, then

$$\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx\,\longrightarrow\,-\frac{2a}{f}\,\mathrm{Subst}\left[\int\frac{1}{2b^2-(a\,c-b\,d)\,x^2}\,dx,\,x,\,\frac{b\,\mathrm{Cos}[e+f\,x]}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\right]$$

Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*a/f*Subst[Int[1/(2*b^2-(a*c-b*d)*x^2),x],x,b*Cos[e+f*x]/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]])] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**8:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n>1$

Derivation: Singly degenerate sine recurrence 2c with $A\to c,\ B\to d,\ n\to n-1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2==0\,\land\,c^2-d^2\neq 0\,\land\,n>1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{d\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-1}}{f\,(m+n)}+$$

$$\frac{1}{b\,(m+n)}\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-2}\,\left(d\,(a\,c\,m+b\,d\,(n-1))+b\,c^2\,(m+n)+(d\,(a\,d\,m+b\,c\,(m+2\,n-1)))\,\mathrm{Sin}[e+f\,x]\right)\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -d*Cos[e+f*x]*(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-1)/(f*(m+n)) +
  1/(b*(m+n))*Int[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^(n-2)*
    Simp[d*(a*c*m+b*d*(n-1))+b*c^2*(m+n)+d*(a*d*m+b*c*(m+2*n-1))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[n,1] && IntegerQ[n]
```

9. $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0$

1: $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m\in\mathbb{Z}$

## Derivation: Piecewise constant extraction and integration by substitution

Basis: $\partial_x \frac{\mathrm{Cos}[e+f\,x]}{\sqrt{1+\mathrm{Sin}[e+f\,x]}\,\sqrt{1-\mathrm{Sin}[e+f\,x]}}==0$

Basis: $\mathrm{Cos}[e+f\,x]==\frac{1}{f}\,\partial_x\,\mathrm{Sin}[e+f\,x]$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2==0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m\in\mathbb{Z}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$

$$\frac{a^m\,\mathrm{Cos}[e+f\,x]}{\sqrt{1+\mathrm{Sin}[e+f\,x]}\,\sqrt{1-\mathrm{Sin}[e+f\,x]}}\int\frac{\mathrm{Cos}[e+f\,x]\,\left(1+\frac{b}{a}\,\mathrm{Sin}[e+f\,x]\right)^{m-\frac{1}{2}}\,(c+d\,\mathrm{Sin}[e+f\,x])^n}{\sqrt{1-\frac{b}{a}\,\mathrm{Sin}[e+f\,x]}}\,dx\,\longrightarrow$$

$$\frac{a^m \operatorname{Cos}[e+f x]}{f \sqrt{1+\operatorname{Sin}[e+f x]}\ \sqrt{1-\operatorname{Sin}[e+f x]}} \operatorname{Subst}\left[\int \frac{\left(1+\frac{b}{a} x\right)^{m-\frac{1}{2}} (c+d x)^n}{\sqrt{1-\frac{b}{a} x}}\, dx,\ x,\ \operatorname{Sin}[e+f x]\right]$$

–

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^m*Cos[e+f*x]/(f*Sqrt[1+Sin[e+f*x]]*Sqrt[1-Sin[e+f*x]])*Subst[Int[(1+b/a*x)^(m-1/2)*(c+d*x)^n/Sqrt[1-b/a*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && IntegerQ[m]
```

2. $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land m \notin \mathbb{Z}$

   1. $\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx$ when $a^2 - b^2 == 0 \land m \notin \mathbb{Z}$

      1. $\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx$ when $a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land a > 0$

         1: $\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx$ when $a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land a > 0 \land \frac{d}{b} > 0$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 == 0$, then $\partial_x \frac{\operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} == 0$

Basis: If $a^2 - b^2 == 0$, then $\frac{b^2 \operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} \frac{\operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} == 1$

Basis: $\frac{\operatorname{Cos}[e+f x] (a+b \operatorname{Sin}[e+f x])^{m-\frac{1}{2}} (b \operatorname{Sin}[e+f x])^n}{\sqrt{a-b \operatorname{Sin}[e+f x]}} ==$

$$-\frac{1}{b f} \operatorname{Subst}\left[\frac{(a-x)^n (2 a-x)^{m-\frac{1}{2}}}{\sqrt{x}}, x, a - b \operatorname{Sin}[e + f x]\right] \partial_x (a - b \operatorname{Sin}[e + f x])$$

Note: If $a > 0$, then $\frac{(a-x)^n (2 a-x)^{m-\frac{1}{2}}}{\sqrt{x}}$ is integrable in terms of the Appell function without the need for additional piecewise constant extraction.

Rule: If $a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land a > 0 \land \frac{d}{b} > 0$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx \longrightarrow$$

$$\frac{b^2 \left(\frac{d}{b}\right)^n \operatorname{Cos}[e + f x]}{\sqrt{a + b \operatorname{Sin}[e + f x]} \sqrt{a - b \operatorname{Sin}[e + f x]}} \int \frac{\operatorname{Cos}[e + f x] (a + b \operatorname{Sin}[e + f x])^{m-\frac{1}{2}} (b \operatorname{Sin}[e + f x])^n}{\sqrt{a - b \operatorname{Sin}[e + f x]}} \, dx \longrightarrow$$

$$-\frac{b\left(\frac{d}{b}\right)^n \operatorname{Cos}[e+f x]}{f\sqrt{a+b\operatorname{Sin}[e+f x]}\,\sqrt{a-b\operatorname{Sin}[e+f x]}}\operatorname{Subst}\left[\int\frac{(a-x)^n\,(2a-x)^{m-\frac{1}{2}}}{\sqrt{x}}\,dx,\ x,\ a-b\operatorname{Sin}[e+f x]\right]$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*(d/b)^n*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*
    Subst[Int[(a-x)^n*(2*a-x)^(m-1/2)/Sqrt[x],x],x,a-b*Sin[e+f*x]] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && GtQ[a,0] && GtQ[d/b,0]
```

**2:** $\int (a+b\operatorname{Sin}[e+f x])^m\,(d\operatorname{Sin}[e+f x])^n\,dx$ when $a^2-b^2=0\ \land\ m\notin\mathbb{Z}\ \land\ a>0\ \land\ \frac{d}{b}\not>0$

## Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(d\operatorname{Sin}[e+f x])^n}{(b\operatorname{Sin}[e+f x])^n}=0$

Rule: If $a^2-b^2=0\ \land\ m\notin\mathbb{Z}\ \land\ a>0\ \land\ \frac{d}{b}\not>0$, then

$$\int (a+b\operatorname{Sin}[e+f x])^m\,(d\operatorname{Sin}[e+f x])^n\,dx\ \longrightarrow\ \frac{\left(\frac{d}{b}\right)^{\operatorname{IntPart}[n]}(d\operatorname{Sin}[e+f x])^{\operatorname{FracPart}[n]}}{(b\operatorname{Sin}[e+f x])^{\operatorname{FracPart}[n]}}\int (a+b\operatorname{Sin}[e+f x])^m\,(b\operatorname{Sin}[e+f x])^n\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  (d/b)^IntPart[n]*(d*Sin[e+f*x])^FracPart[n]/(b*Sin[e+f*x])^FracPart[n]*Int[(a+b*Sin[e+f*x])^m*(b*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && GtQ[a,0] && Not[GtQ[d/b,0]]
```

**2:** $\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx$ when $a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land a \ngtr 0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(a+b \operatorname{Sin}[e+f x])^m}{\left(1+\frac{b}{a} \operatorname{Sin}[e+f x]\right)^m} == 0$

Rule: If $a^2 - b^2 == 0 \land m \notin \mathbb{Z} \land a \ngtr 0$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m (d \operatorname{Sin}[e + f x])^n \, dx \to \frac{a^{\operatorname{IntPart}[m]} (a + b \operatorname{Sin}[e + f x])^{\operatorname{FracPart}[m]}}{\left(1 + \frac{b}{a} \operatorname{Sin}[e + f x]\right)^{\operatorname{FracPart}[m]}} \int \left(1 + \frac{b}{a} \operatorname{Sin}[e + f x]\right)^m (d \operatorname{Sin}[e + f x])^n \, dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^IntPart[m]*(a+b*Sin[e+f*x])^FracPart[m]/(1+b/a*Sin[e+f*x])^FracPart[m]*
    Int[(1+b/a*Sin[e+f*x])^m*(d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,d,e,f,m,n},x] && EqQ[a^2-b^2,0] && Not[IntegerQ[m]] && Not[GtQ[a,0]]
```

**2:** $\int (a + b \operatorname{Sin}[e + f x])^m (c + d \operatorname{Sin}[e + f x])^n \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land m \notin \mathbb{Z}$

Derivation: Piecewise constant extraction and integration by substitution

Basis: If $a^2 - b^2 == 0$, then $\partial_x \frac{\operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} == 0$

Basis: If $a^2 - b^2 == 0$, then $\frac{a^2 \operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} \frac{\operatorname{Cos}[e+f x]}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{a-b \operatorname{Sin}[e+f x]}} == 1$

Basis: $\operatorname{Cos}[e + f x] == \frac{1}{f} \partial_x \operatorname{Sin}[e + f x]$

Rule: If $b c - a d \neq 0 \land a^2 - b^2 == 0 \land c^2 - d^2 \neq 0 \land m \notin \mathbb{Z}$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx\;\longrightarrow$$

$$\frac{a^2\,\text{Cos}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{a-b\,\text{Sin}[e+f\,x]}}\int\frac{\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-\frac{1}{2}}\,(c+d\,\text{Sin}[e+f\,x])^n}{\sqrt{a-b\,\text{Sin}[e+f\,x]}}\,dx\;\longrightarrow$$

$$\frac{a^2\,\text{Cos}[e+f\,x]}{f\,\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{a-b\,\text{Sin}[e+f\,x]}}\,\text{Subst}\Big[\int\frac{(a+b\,x)^{m-\frac{1}{2}}\,(c+d\,x)^n}{\sqrt{a-b\,x}}\,dx,\,x,\,\text{Sin}[e+f\,x]\Big]$$

–

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  a^2*Cos[e+f*x]/(f*Sqrt[a+b*Sin[e+f*x]]*Sqrt[a-b*Sin[e+f*x]])*Subst[Int[(a+b*x)^(m-1/2)*(c+d*x)^n/Sqrt[a-b*x],x],x,Sin[e+f*x]] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[b*c-a*d,0] && EqQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[IntegerQ[m]]
```

8. $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x])^n \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land c^2 - d^2 \neq 0$

1. $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x])^2 \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0$

1: $\int (b \sin[e + f x])^m (c + d \sin[e + f x])^2 \, dx$

Derivation: Algebraic expansion

Basis: $(c + d z)^2 == \frac{2 c d}{b} (b z) + (c^2 + d^2 z^2)$

Rule:

$$\int (b \sin[e + f x])^m (c + d \sin[e + f x])^2 \, dx \to$$
$$\frac{2 c d}{b} \int (b \sin[e + f x])^{m+1} \, dx + \int (b \sin[e + f x])^m (c^2 + d^2 \sin[e + f x]^2) \, dx$$

Program code:

```
Int[(b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  2*c*d/b*Int[(b*Sin[e+f*x])^(m+1),x] + Int[(b*Sin[e+f*x])^m*(c^2+d^2*Sin[e+f*x]^2),x] /;
FreeQ[{b,c,d,e,f,m},x]
```

2: $\int (a + b \sin[e + f x])^m (c + d \sin[e + f x])^2 \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land m < -1$

Derivation: Nondegenerate sine recurrence 1a with $A \to c^2$, $B \to 2 c d$, $C \to d^2$, $n \to 0$, $p \to 0$

Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land m < -1$, then

$$\int (a + b \sin[e + f x])^m (c + d \sin[e + f x])^2 \, dx \to$$
$$-\frac{(b^2 c^2 - 2 a b c d + a^2 d^2) \cos[e + f x] (a + b \sin[e + f x])^{m+1}}{b f (m + 1) (a^2 - b^2)} -$$

$$\frac{1}{b\,(m+1)\,(a^2-b^2)}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m+1}\,\left(b\,(m+1)\,\left(2\,b\,c\,d-a\,(c^2+d^2)\right)+\left(a^2\,d^2-2\,a\,b\,c\,d\,(m+2)+b^2\,\left(d^2\,(m+1)+c^2\,(m+2)\right)\right)\,\mathrm{Sin}[e+f\,x]\right)\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  -(b^2*c^2-2*a*b*c*d+a^2*d^2)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+1)*(a^2-b^2)) -
  1/(b*(m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*
     Simp[b*(m+1)*(2*b*c*d-a*(c^2+d^2))+(a^2*d^2-2*a*b*c*d*(m+2)+b^2*(d^2*(m+1)+c^2*(m+2)))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && LtQ[m,-1]
```

**3:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,m\not<-1$

## Derivation: Nondegenerate sine recurrence 1b with $A\to a^2,\ B\to 2\,a\,b,\ C\to b^2,\ m\to 0,\ p\to 0$

## Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,m\not<-1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^2\,dx\,\longrightarrow$$

$$-\frac{d^2\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^{m+1}}{b\,f\,(m+2)}+$$

$$\frac{1}{b\,(m+2)}\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,\left(b\,\left(d^2\,(m+1)+c^2\,(m+2)\right)-d\,(a\,d-2\,b\,c\,(m+2))\,\mathrm{Sin}[e+f\,x]\right)\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^2,x_Symbol] :=
  -d^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)/(b*f*(m+2)) +
  1/(b*(m+2))*Int[(a+b*Sin[e+f*x])^m*Simp[b*(d^2*(m+1)+c^2*(m+2))-d*(a*d-2*b*c*(m+2))*Sin[e+f*x],x],x] /;
FreeQ[{a,b,c,d,e,f,m},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && Not[LtQ[m,-1]]
```

**x:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n\,dx$ when $a^2-b^2\neq 0\,\land\,m\in\mathbb{Z}^+$

Derivation: Algebraic expansion

Note: If terms having the same powers of $\text{sin}[e+f\,x]$ are collected, this rule results in more compact antiderivatives; however, the number of steps required grows exponentially with $m$.

Rule: If $a^2-b^2\neq 0\,\land\,m\in\mathbb{Z}^+$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n\,dx\,\rightarrow\,\int \text{ExpandTrig}\left[(a+b\,\text{Sin}[e+f\,x])^m\,(d\,\text{Sin}[e+f\,x])^n,\,x\right]dx$$

Program code:

```
(* Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(a+b*sin[e+f*x])^m*(d*sin[e+f*x])^n,x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && IGtQ[m,0] *)
```

**2.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,m>2$

**1:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,m>2\,\land\,n<-1$

Derivation: Nondegenerate sine recurrence 1a with $A\to c^2,\ B\to 2\,c\,d,\ C\to d^2,\ n\to n-2,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,m>2\,\land\,n<-1$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx\,\rightarrow$$
$$-\left(\left(\left(b^2\,c^2-2\,a\,b\,c\,d+a^2\,d^2\right)\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}\right)\Big/\left(d\,f\,(n+1)\,\left(c^2-d^2\right)\right)\right)+$$
$$\frac{1}{d\,(n+1)\,\left(c^2-d^2\right)}\int (a+b\,\text{Sin}[e+f\,x])^{m-3}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}\cdot$$
$$\left(b\,(m-2)\,(b\,c-a\,d)^2+a\,d\,(n+1)\,\left(c\,\left(a^2+b^2\right)-2\,a\,b\,d\right)+\right.$$
$$\left(b\,(n+1)\,\left(a\,b\,c^2+c\,d\,\left(a^2+b^2\right)-3\,a\,b\,d^2\right)-a\,(n+2)\,(b\,c-a\,d)^2\right)\,\text{Sin}[e+f\,x]\,+$$

$$b\left(b^2\left(c^2-d^2\right)-m\left(b\,c-a\,d\right)^2+d\,n\left(2\,a\,b\,c-d\left(a^2+b^2\right)\right)\right)\mathrm{Sin}\left[e+f\,x\right]^2\right)\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -(b^2*c^2-2*a*b*c*d+a^2*d^2)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(n+1)*(c^2-d^2)) +
  1/(d*(n+1)*(c^2-d^2))*Int[(a+b*Sin[e+f*x])^(m-3)*(c+d*Sin[e+f*x])^(n+1)*
    Simp[b*(m-2)*(b*c-a*d)^2+a*d*(n+1)*(c*(a^2+b^2)-2*a*b*d)+
      (b*(n+1)*(a*b*c^2+c*d*(a^2+b^2)-3*a*b*d^2)-a*(n+2)*(b*c-a*d)^2)*Sin[e+f*x]+
      b*(b^2*(c^2-d^2)-m*(b*c-a*d)^2+d*n*(2*a*b*c-d*(a^2+b^2)))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,2] && LtQ[n,-1] && (IntegerQ[m] || IntegersQ[2*m,2*n])
```

**2:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m>2\,\wedge\,n\not<-1$

Derivation: Nondegenerate sine recurrence 1b with $A\to a^2$, $B\to 2\,a\,b$, $C\to b^2$, $m\to m-2$, $p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m>2\,\wedge\,n\not<-1$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{b^2\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m-2}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}}{d\,f\,(m+n)}+$$
$$\frac{1}{d\,(m+n)}\int (a+b\,\text{Sin}[e+f\,x])^{m-3}\,(c+d\,\text{Sin}[e+f\,x])^n\,\cdot$$
$$\left(a^3\,d\,(m+n)+b^2\,(b\,c\,(m-2)+a\,d\,(n+1))-b\,\left(a\,b\,c-b^2\,d\,(m+n-1)-3\,a^2\,d\,(m+n)\right)\,\text{Sin}[e+f\,x]-b^2\,(b\,c\,(m-1)-a\,d\,(3\,m+2\,n-2))\,\text{Sin}[e+f\,x]^2\right)\,dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n+1)/(d*f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-3)*(c+d*Sin[e+f*x])^n*
    Simp[a^3*d*(m+n)+b^2*(b*c*(m-2)+a*d*(n+1))-
      b*(a*b*c-b^2*d*(m+n-1)-3*a^2*d*(m+n))*Sin[e+f*x]-
      b^2*(b*c*(m-1)-a*d*(3*m+2*n-2))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[m,2] &&
  (IntegerQ[m] || IntegersQ[2*m,2*n]) && Not[IGtQ[n,2] && (Not[IntegerQ[m]] || EqQ[a,0] && NeQ[c,0])]
```

**3.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1$

**1.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,0<n<2$

**1.** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,0<n<1$

**1.** $\int \frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{(a+b\,\text{Sin}[e+f\,x])^{3/2}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

**1:** $\int \frac{\sqrt{d \sin[e+f x]}}{(a+b \sin[e+f x])^{3/2}} \, dx$ when $a^2-b^2 \neq 0$

Derivation: Nondegenerate sine recurrence 1a with $A \to 0$, $B \to d$, $C \to 0$, $m \to -\frac{3}{2}$, $n \to -\frac{1}{2}$, $p \to 0$

Rule: If $a^2-b^2 \neq 0$, then

$$\int \frac{\sqrt{d \sin[e+f x]}}{(a+b \sin[e+f x])^{3/2}} \, dx \to -\frac{2 a d \cos[e+f x]}{f (a^2-b^2) \sqrt{a+b \sin[e+f x]} \sqrt{d \sin[e+f x]}} - \frac{d^2}{a^2-b^2} \int \frac{\sqrt{a+b \sin[e+f x]}}{(d \sin[e+f x])^{3/2}} \, dx$$

Program code:

```
Int[Sqrt[d_.*sin[e_.+f_.*x_]]/(a_+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  -2*a*d*Cos[e+f*x]/(f*(a^2-b^2)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]) -
  d^2/(a^2-b^2)*Int[Sqrt[a+b*Sin[e+f*x]]/(d*Sin[e+f*x])^(3/2),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

**2:** $\int \frac{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx$ when $b\,c-a\,d\neq 0\ \wedge\ a^2-b^2\neq 0\ \wedge\ c^2-d^2\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{\sqrt{c+d\,z}}{(a+b\,z)^{3/2}} == \frac{c-d}{a-b}\,\frac{1}{\sqrt{a+b\,z}\,\sqrt{c+d\,z}} - \frac{b\,c-a\,d}{a-b}\,\frac{1+z}{(a+b\,z)^{3/2}\,\sqrt{c+d\,z}}$

Rule: If $b\,c-a\,d\neq 0\ \wedge\ a^2-b^2\neq 0\ \wedge\ c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx \rightarrow$$

$$\frac{c-d}{a-b}\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx - \frac{b\,c-a\,d}{a-b}\int \frac{1+\mathrm{Sin}[e+f\,x]}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[Sqrt[c_+d_.*sin[e_.+f_.*x_]]/(a_.+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  (c-d)/(a-b)*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] -
  (b*c-a*d)/(a-b)*Int[(1+Sin[e+f*x])/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\ \wedge\ a^2-b^2\neq 0\ \wedge\ c^2-d^2\neq 0\ \wedge\ m<-1\ \wedge\ 0<n<1$

## Derivation: Nondegenerate sine recurrence 1a with $A\to 1,\ B\to 0,\ C\to 0,\ p\to 0$

## Derivation: Nondegenerate sine recurrence 1c with $A\to c,\ B\to d,\ C\to 0,\ n\to n-1,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\ \wedge\ a^2-b^2\neq 0\ \wedge\ c^2-d^2\neq 0\ \wedge\ m<-1\ \wedge\ 0<n<1$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx \rightarrow$$

$$-\frac{b \operatorname{Cos}[e+f x] (a+b \operatorname{Sin}[e+f x])^{m+1} (c+d \operatorname{Sin}[e+f x])^{n}}{f (m+1) (a^{2}-b^{2})}+$$
$$\frac{1}{(m+1) (a^{2}-b^{2})} \int (a+b \operatorname{Sin}[e+f x])^{m+1} (c+d \operatorname{Sin}[e+f x])^{n-1} \cdot$$
$$(a c (m+1)+b d n+(a d (m+1)-b c (m+2)) \operatorname{Sin}[e+f x]-b d (m+n+2) \operatorname{Sin}[e+f x]^{2}) \, dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)*
    Simp[a*c*(m+1)+b*d*n+(a*d*(m+1)-b*c*(m+2))*Sin[e+f*x]-b*d*(m+n+2)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[0,n,1] && IntegersQ[2*m,2*n]
```

2. $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,1<n<2$

1. $\int \frac{(c+d\,\mathrm{Sin}[e+f\,x])^{3/2}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

1: $\int \frac{(d\,\mathrm{Sin}[e+f\,x])^{3/2}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx$ when $a^2-b^2\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{(d\,z)^{3/2}}{(a+b\,z)^{3/2}}==\frac{d\,\sqrt{d\,z}}{b\,\sqrt{a+b\,z}}-\frac{a\,d\,\sqrt{d\,z}}{b\,(a+b\,z)^{3/2}}$

Rule: If $a^2-b^2\neq 0$, then

$$\int \frac{(d\,\mathrm{Sin}[e+f\,x])^{3/2}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx\,\rightarrow\,\frac{d}{b}\int \frac{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\,dx-\frac{a\,d}{b}\int \frac{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx$$

## Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^(3/2)/(a_+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  d/b*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] -
  a*d/b*Int[Sqrt[d*Sin[e+f*x]]/(a+b*Sin[e+f*x])^(3/2),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

**2:** $\int \frac{(c+d\,\mathrm{Sin}[e+f\,x])^{3/2}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{(c+d\,z)^{3/2}}{(a+b\,z)^{3/2}}==\frac{d^2\sqrt{a+b\,z}}{b^2\sqrt{c+d\,z}}+\frac{(b\,c-a\,d)\,(b\,c+a\,d+2\,b\,d\,z)}{b^2\,(a+b\,z)^{3/2}\sqrt{c+d\,z}}$

## Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$, then

$$\int \frac{(c+d\,\mathrm{Sin}[e+f\,x])^{3/2}}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}}\,dx\,\longrightarrow\,\frac{d^2}{b^2}\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx+\frac{(b\,c-a\,d)}{b^2}\int \frac{b\,c+a\,d+2\,b\,d\,\mathrm{Sin}[e+f\,x]}{(a+b\,\mathrm{Sin}[e+f\,x])^{3/2}\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[(c_+d_.*sin[e_.+f_.*x_])^(3/2)/(a_.+b_.*sin[e_.+f_.*x_])^(3/2),x_Symbol] :=
  d^2/b^2*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b^2*Int[Simp[b*c+a*d+2*b*d*Sin[e+f*x],x]/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,1<n<2$

## Derivation: Nondegenerate sine recurrence 1a with $A\to c,\ B\to d,\ C\to 0,\ n\to n-1,\ p\to 0$

## Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,1<n<2$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{(b\,c-a\,d)\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^{m+1}\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-1}}{f\,(m+1)\,(a^2-b^2)}+$$
$$\frac{1}{(m+1)\,(a^2-b^2)}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m+1}\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-2}\cdot$$

$$\left(c\,(a\,c-b\,d)\,(m+1)+d\,(b\,c-a\,d)\,(n-1)+(d\,(a\,c-b\,d)\,(m+1)-c\,(b\,c-a\,d)\,(m+2))\,\text{Sin}[e+f\,x]-d\,(b\,c-a\,d)\,(m+n+1)\,\text{Sin}[e+f\,x]^2\right)dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -(b*c-a*d)*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-1)/(f*(m+1)*(a^2-b^2)) +
  1/((m+1)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n-2)*
    Simp[c*(a*c-b*d)*(m+1)+d*(b*c-a*d)*(n-1)+(d*(a*c-b*d)*(m+1)-c*(b*c-a*d)*(m+2))*Sin[e+f*x]-d*(b*c-a*d)*(m+n+1)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && LtQ[1,n,2] && IntegersQ[2*m,2*n]
```

2. $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n\not>0$

1. $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

1: $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0$

## Derivation: Nondegenerate sine recurrence 1a with $c\to 0,\ A\to 1,\ B\to 0,\ C\to 0,\ p\to 0,\ m\to -\frac{3}{2},\ n\to -\frac{1}{2}$

## Rule: If $a^2-b^2\neq 0$, then

$$\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx\to\frac{2\,b\,\text{Cos}[e+f\,x]}{f\,(a^2-b^2)\,\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}+\frac{d}{a^2-b^2}\int \frac{b+a\,\text{Sin}[e+f\,x]}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,(d\,\text{Sin}[e+f\,x])^{3/2}}\,dx$$

## Program code:

```
Int[1/((a_+b_.*sin[e_.+f_.*x_])^(3/2)*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2*b*Cos[e+f*x]/(f*(a^2-b^2)*Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]) +
  d/(a^2-b^2)*Int[(b+a*Sin[e+f*x])/(Sqrt[a+b*Sin[e+f*x]]*(d*Sin[e+f*x])^(3/2)),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

**2:** $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

Derivation: Algebraic expansion

Basis: $\frac{1}{(a+b\,z)^{3/2}} == \frac{1}{(a-b)\,\sqrt{a+b\,z}} - \frac{b\,(1+z)}{(a-b)\,(a+b\,z)^{3/2}}$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$, then

$$\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \rightarrow$$
$$\frac{1}{a-b}\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx-\frac{b}{a-b}\int \frac{1+\text{Sin}[e+f\,x]}{(a+b\,\text{Sin}[e+f\,x])^{3/2}\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])^(3/2)*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  1/(a-b)*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] -
  b/(a-b)*Int[(1+Sin[e+f*x])/((a+b*Sin[e+f*x])^(3/2)*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**2:** $\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n\not>0$

Derivation: Nondegenerate sine recurrence 1c with $A\to 1,\ B\to 0,\ C\to 0,\ p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,m<-1\,\wedge\,n\not>0$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx \rightarrow$$
$$-\frac{b^2\,\text{Cos}[e+f\,x]\,(a+b\,\text{Sin}[e+f\,x])^{m+1}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}}{f\,(m+1)\,(b\,c-a\,d)\,(a^2-b^2)}+$$
$$\frac{1}{(m+1)\,(b\,c-a\,d)\,(a^2-b^2)}\int (a+b\,\text{Sin}[e+f\,x])^{m+1}\,(c+d\,\text{Sin}[e+f\,x])^n\cdot$$

$$\left(a\,(b\,c-a\,d)\,(m+1)+b^2\,d\,(m+n+2)-\left(b^2\,c+b\,(b\,c-a\,d)\,(m+1)\right)\,\text{Sin}[e+f\,x]-b^2\,d\,(m+n+3)\,\text{Sin}[e+f\,x]^2\right)\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b^2*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^(n+1)/(f*(m+1)*(b*c-a*d)*(a^2-b^2)) +
  1/((m+1)*(b*c-a*d)*(a^2-b^2))*Int[(a+b*Sin[e+f*x])^(m+1)*(c+d*Sin[e+f*x])^n*
    Simp[a*(b*c-a*d)*(m+1)+b^2*d*(m+n+2)-(b^2*c+b*(b*c-a*d)*(m+1))*Sin[e+f*x]-b^2*d*(m+n+3)*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[m,-1] && IntegersQ[2*m,2*n] &&
  (EqQ[a,0] && IntegerQ[m] && Not[IntegerQ[n]] || Not[IntegerQ[2*n] && LtQ[n,-1] && (IntegerQ[n] && Not[IntegerQ[m]] || EqQ[a,0])])
```

4: $\int \frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{a+b\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0$

## Derivation: Algebraic expansion

## Basis: $\frac{\sqrt{c+d\,z}}{a+b\,z}==\frac{d}{b\,\sqrt{c+d\,z}}+\frac{b\,c-a\,d}{b\,(a+b\,z)\,\sqrt{c+d\,z}}$

## Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0$, then

$$\int \frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{a+b\,\text{Sin}[e+f\,x]}\,dx\,\longrightarrow\,\frac{d}{b}\int \frac{1}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx+\frac{b\,c-a\,d}{b}\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[Sqrt[c_.+d_.*sin[e_.+f_.*x_]]/(a_.+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  d/b*Int[1/Sqrt[c+d*Sin[e+f*x]],x] +
  (b*c-a*d)/b*Int[1/((a+b*Sin[e+f*x])*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

**5:** $\int \frac{(a+b\,\text{Sin}[e+f\,x])^{3/2}}{c+d\,\text{Sin}[e+f\,x]}\,dx$ when $b\,c-a\,d\neq 0\wedge a^2-b^2\neq 0\wedge c^2-d^2\neq 0$

## Derivation: Algebraic expansion

Basis: $\frac{a+b\,z}{c+d\,z}=\frac{b}{d}-\frac{b\,c-a\,d}{d\,(c+d\,z)}$

Rule: If $b\,c-a\,d\neq 0\wedge a^2-b^2\neq 0\wedge c^2-d^2\neq 0$, then

$$\int \frac{(a+b\,\text{Sin}[e+f\,x])^{3/2}}{c+d\,\text{Sin}[e+f\,x]}\,dx\;\to\;\frac{b}{d}\int\sqrt{a+b\,\text{Sin}[e+f\,x]}\,dx-\frac{b\,c-a\,d}{d}\int\frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{c+d\,\text{Sin}[e+f\,x]}\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^(3/2)/(c_.+d_.*sin[e_.+f_.*x_]),x_Symbol] :=
  b/d*Int[Sqrt[a+b*Sin[e+f*x]],x] - (b*c-a*d)/d*Int[Sqrt[a+b*Sin[e+f*x]]/(c+d*Sin[e+f*x]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

6. $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

1: $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,c+d>0$

Derivation: Primitive rule

Basis: If $c+d>0$, then $\partial_x\,\text{EllipticPi}\left[\frac{2\,b}{a+b},\,\frac{1}{2}\left(x-\frac{\pi}{2}\right),\,\frac{2\,d}{c+d}\right]==\frac{(a+b)\,\sqrt{c+d}}{2\,(a+b\,\text{Sin}[x])\,\sqrt{c+d\,\text{Sin}[x]}}$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,c+d>0$, then

$$\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx\;\rightarrow\;\frac{2}{f\,(a+b)\,\sqrt{c+d}}\,\text{EllipticPi}\left[\frac{2\,b}{a+b},\,\frac{1}{2}\left(e-\frac{\pi}{2}+f\,x\right),\,\frac{2\,d}{c+d}\right]$$

Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2/(f*(a+b)*Sqrt[c+d])*EllipticPi[2*b/(a+b),1/2*(e-Pi/2+f*x),2*d/(c+d)] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[c+d,0]
```

2: $\int \frac{1}{(a+b\,\text{Sin}[e+f\,x])\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,c-d>0$

Derivation: Primitive rule

Basis: If $c-d>0$, then $\partial_x\,\text{EllipticPi}\left[-\frac{2\,b}{a-b},\,\frac{1}{2}\left(x+\frac{\pi}{2}\right),\,-\frac{2\,d}{c-d}\right]==\frac{(a-b)\,\sqrt{c-d}}{2\,(a+b\,\text{Sin}[x])\,\sqrt{c+d\,\text{Sin}[x]}}$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,c-d>0$, then

$$\int \frac{1}{(a + b \operatorname{Sin}[e + f x]) \sqrt{c + d \operatorname{Sin}[e + f x]}} \, dx \to \frac{2}{f\,(a - b)\,\sqrt{c - d}} \operatorname{EllipticPi}\left[-\frac{2 b}{a - b}, \frac{1}{2}\left(e + \frac{\pi}{2} + f x\right), -\frac{2 d}{c - d}\right]$$

## Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2/(f*(a-b)*Sqrt[c-d])*EllipticPi[-2*b/(a-b),1/2*(e+Pi/2+f*x),-2*d/(c-d)] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && GtQ[c-d,0]
```

**3:** $\int \frac{1}{(a + b \operatorname{Sin}[e + f x]) \sqrt{c + d \operatorname{Sin}[e + f x]}} \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land c^2 - d^2 \neq 0 \land c + d \ngtr 0$

## Derivation: Piecewise constant extraction

## Basis: $\partial_x \frac{\sqrt{\frac{c+d F[x]}{c+d}}}{\sqrt{c+d F[x]}} == 0$

## Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land c^2 - d^2 \neq 0 \land c + d \ngtr 0$, then

$$\int \frac{1}{(a + b \operatorname{Sin}[e + f x]) \sqrt{c + d \operatorname{Sin}[e + f x]}} \, dx \to \frac{\sqrt{\frac{c+d \operatorname{Sin}[e+f x]}{c+d}}}{\sqrt{c + d \operatorname{Sin}[e + f x]}} \int \frac{1}{(a + b \operatorname{Sin}[e + f x]) \sqrt{\frac{c}{c+d} + \frac{d}{c+d} \operatorname{Sin}[e + f x]}} \, dx$$

## Program code:

```
Int[1/((a_.+b_.*sin[e_.+f_.*x_])*Sqrt[c_.+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[(c+d*Sin[e+f*x])/(c+d)]/Sqrt[c+d*Sin[e+f*x]]*Int[1/((a+b*Sin[e+f*x])*Sqrt[c/(c+d)+d/(c+d)*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && Not[GtQ[c+d,0]]
```

7. $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\land a^2-b^2\neq 0\land c^2-d^2\neq 0$

1. $\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2-d^2\neq 0$

1. $\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2-d^2\neq 0\land \frac{c+d}{b}>0$

1: $\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2-d^2>0\land \frac{c+d}{b}>0\land c^2>0$

## Rule: If $c^2-d^2>0\land \frac{c+d}{b}>0\land c^2>0$, then

$$\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx\to$$

$$\frac{2\,c\,\sqrt{b\,(c+d)}\,\text{Tan}[e+f\,x]\,\sqrt{1+\text{Csc}[e+f\,x]}\,\sqrt{1-\text{Csc}[e+f\,x]}}{d\,f\,\sqrt{c^2-d^2}}\,\text{EllipticPi}\left[\frac{c+d}{d},\,\text{ArcSin}\left[\frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{\sqrt{b\,\text{Sin}[e+f\,x]}}\Big/\sqrt{\frac{c+d}{b}}\right],\,-\frac{c+d}{c-d}\right]$$

## Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*c*Rt[b*(c+d),2]*Tan[e+f*x]*Sqrt[1+Csc[e+f*x]]*Sqrt[1-Csc[e+f*x]]/(d*f*Sqrt[c^2-d^2])*
    EllipticPi[(c+d)/d,ArcSin[Sqrt[c+d*Sin[e+f*x]]/Sqrt[b*Sin[e+f*x]]/Rt[(c+d)/b,2]],-(c+d)/(c-d)] /;
FreeQ[{b,c,d,e,f},x] && GtQ[c^2-d^2,0] && PosQ[(c+d)/b] && GtQ[c^2,0]
```

2: $\int \frac{\sqrt{b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $c^2-d^2\neq 0\land \frac{c+d}{b}>0$

## Rule: If $c^2-d^2\neq 0\land \frac{c+d}{b}>0$, then

$$\int \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx \to$$

$$\frac{2\,b\,\mathrm{Tan}[e+f\,x]}{d\,f}\sqrt{\frac{c+d}{b}}\sqrt{\frac{c\,(1+\mathrm{Csc}[e+f\,x])}{c-d}}\sqrt{\frac{c\,(1-\mathrm{Csc}[e+f\,x])}{c+d}}\,\mathrm{EllipticPi}\Big[\frac{c+d}{d},\,\mathrm{ArcSin}\Big[\frac{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}\Big/\sqrt{\frac{c+d}{b}}\Big],\,-\frac{c+d}{c-d}\Big]$$

## Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*b*Tan[e+f*x]/(d*f)*Rt[(c+d)/b,2]*Sqrt[c*(1+Csc[e+f*x])/(c-d)]*Sqrt[c*(1-Csc[e+f*x])/(c+d)]*
    EllipticPi[(c+d)/d,ArcSin[Sqrt[c+d*Sin[e+f*x]]/Sqrt[b*Sin[e+f*x]]/Rt[(c+d)/b,2]],-(c+d)/(c-d)] /;
FreeQ[{b,c,d,e,f},x] && NeQ[c^2-d^2,0] && PosQ[(c+d)/b]
```

**2:** $\int \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $c^2-d^2\neq 0\,\wedge\,\frac{c+d}{b}\ngtr 0$

## Derivation: Piecewise constant extraction

## Basis: $\partial_x \frac{\sqrt{F[x]}}{\sqrt{-F[x]}} == 0$

## Rule: If $c^2-d^2\neq 0\,\wedge\,\frac{c+d}{b}\ngtr 0$, then

$$\int \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx \to \frac{\sqrt{b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{-b\,\mathrm{Sin}[e+f\,x]}}\int \frac{\sqrt{-b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[Sqrt[b_.*sin[e_.+f_.*x_]]/Sqrt[c_+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  Sqrt[b*Sin[e+f*x]]/Sqrt[-b*Sin[e+f*x]]*Int[Sqrt[-b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]],x] /;
FreeQ[{b,c,d,e,f},x] && NeQ[c^2-d^2,0] && NegQ[(c+d)/b]
```

**2.** $\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\ \land\ a^2-b^2\neq 0\ \land\ c^2-d^2\neq 0$

**x:** $\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0$

Derivation: Algebraic expansion

Basis: $\frac{\sqrt{a+b\,z}}{\sqrt{d\,z}} == \frac{a}{\sqrt{a+b\,z}\,\sqrt{d\,z}}+\frac{b\,\sqrt{d\,z}}{d\,\sqrt{a+b\,z}}$

Rule: If $a^2-b^2\neq 0$, then

$$\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx \to a\int \frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx+\frac{b}{d}\int \frac{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\,dx$$

Program code:

```
(* Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  a*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[d*Sin[e+f*x]]),x] +
  b/d*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] *)
```

**x:** $\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0\ \land\ \frac{a+b}{d}>0$

Rule: If $a^2-b^2\neq 0\ \land\ \frac{a+b}{d}>0$, then

$$\int \frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}\,dx \to$$

$$\frac{2\,(a+b\,\mathrm{Sin}[e+f\,x])}{d\,f\,\sqrt{\frac{a+b}{d}}\,\mathrm{Cos}[e+f\,x]}\sqrt{\frac{a\,(1-\mathrm{Sin}[e+f\,x])}{a+b\,\mathrm{Sin}[e+f\,x]}}\sqrt{\frac{a\,(1+\mathrm{Sin}[e+f\,x])}{a+b\,\mathrm{Sin}[e+f\,x]}}\,\mathrm{EllipticPi}\Big[\frac{b}{a+b},\,\mathrm{ArcSin}\Big[\sqrt{\frac{a+b}{d}}\,\frac{\sqrt{d\,\mathrm{Sin}[e+f\,x]}}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}\Big],\,-\frac{a-b}{a+b}\Big]$$

–

## Program code:

```
(* Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*(a+b*Sin[e+f*x])/(d*f*Rt[(a+b)/d,2]*Cos[e+f*x])*Sqrt[a*(1-Sin[e+f*x])/(a+b*Sin[e+f*x])]*Sqrt[a*(1+Sin[e+f*x])/(a+b*Sin[e+f*x])]*
    EllipticPi[b/(a+b),ArcSin[Rt[(a+b)/d,2]*(Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]])],-(a-b)/(a+b)] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && PosQ[(a+b)/d] *)
```

**1:** $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,\frac{a+b}{c+d}>0$

Rule: If $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,\frac{a+b}{c+d}>0$, then

$$\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \,\longrightarrow$$

$$\frac{2\,(a+b\,\text{Sin}[e+f\,x])}{d\,f\,\sqrt{\frac{a+b}{c+d}}\,\text{Cos}[e+f\,x]}\sqrt{\frac{(b\,c-a\,d)\,(1+\text{Sin}[e+f\,x])}{(c-d)\,(a+b\,\text{Sin}[e+f\,x])}}$$

$$\sqrt{-\frac{(b\,c-a\,d)\,(1-\text{Sin}[e+f\,x])}{(c+d)\,(a+b\,\text{Sin}[e+f\,x])}}\,\text{EllipticPi}\Big[\frac{b\,(c+d)}{d\,(a+b)},\,\text{ArcSin}\Big[\sqrt{\frac{a+b}{c+d}}\,\frac{\sqrt{c+d\,\text{Sin}[e+f\,x]}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\Big],\,\frac{(a-b)\,(c+d)}{(a+b)\,(c-d)}\Big]$$

Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  2*(a+b*Sin[e+f*x])/(d*f*Rt[(a+b)/(c+d),2]*Cos[e+f*x])*
    Sqrt[(b*c-a*d)*(1+Sin[e+f*x])/((c-d)*(a+b*Sin[e+f*x]))]*
    Sqrt[-(b*c-a*d)*(1-Sin[e+f*x])/((c+d)*(a+b*Sin[e+f*x]))]*
    EllipticPi[b*(c+d)/(d*(a+b)),ArcSin[Rt[(a+b)/(c+d),2]*Sqrt[c+d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]]],(a-b)*(c+d)/((a+b)*(c-d))] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && PosQ[(a+b)/(c+d)]
```

**2:** $\int \frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,\frac{a+b}{c+d}\not>0$

Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{F[x]}}{\sqrt{-F[x]}} == 0$

## Rule: If $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land c^2 - d^2 \neq 0 \land \frac{a+b}{c+d} \ngtr 0$, then

$$\int \frac{\sqrt{a + b \operatorname{Sin}[e + f x]}}{\sqrt{c + d \operatorname{Sin}[e + f x]}} \, dx \to \frac{\sqrt{-c - d \operatorname{Sin}[e + f x]}}{\sqrt{c + d \operatorname{Sin}[e + f x]}} \int \frac{\sqrt{a + b \operatorname{Sin}[e + f x]}}{\sqrt{-c - d \operatorname{Sin}[e + f x]}} \, dx$$

## Program code:

```
Int[Sqrt[a_+b_.*sin[e_.+f_.*x_]]/Sqrt[c_.+d_.*sin[e_.+f_.*x_]],x_Symbol] :=
  Sqrt[-c-d*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]]*Int[Sqrt[a+b*Sin[e+f*x]]/Sqrt[-c-d*Sin[e+f*x]],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && NegQ[(a+b)/(c+d)]
```

8. $\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f x]} \sqrt{c + d \operatorname{Sin}[e + f x]}} \, dx$ when $b c - a d \neq 0 \land a^2 - b^2 \neq 0 \land c^2 - d^2 \neq 0$

   1. $\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f x]} \sqrt{d \operatorname{Sin}[e + f x]}} \, dx$ when $a^2 - b^2 \neq 0$

      1. $\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f x]} \sqrt{d \operatorname{Sin}[e + f x]}} \, dx$ when $a^2 - b^2 < 0 \land b^2 > 0$

         1: $\int \frac{1}{\sqrt{a + b \operatorname{Sin}[e + f x]} \sqrt{d \operatorname{Sin}[e + f x]}} \, dx$ when $a^2 - b^2 < 0 \land d^2 == 1 \land b d > 0$

## Derivation: Integration by substitution

## Basis: If $a^2 - b^2 < 0 \land d^2 == 1 \land b d > 0$, then

$$\frac{1}{\sqrt{a+b \operatorname{Sin}[e+f x]} \sqrt{d \operatorname{Sin}[e+f x]}} == -\frac{2 d}{f \sqrt{a+b d}} \operatorname{Subst}\left[\frac{1}{\sqrt{1-x^2} \sqrt{1+\frac{(a-b d) x^2}{a+b d}}}, x, \frac{\operatorname{Cos}[e+f x]}{1+d \operatorname{Sin}[e+f x]}\right] \partial_x \frac{\operatorname{Cos}[e+f x]}{1+d \operatorname{Sin}[e+f x]}$$

## Rule: If $a^2 - b^2 < 0 \land d^2 == 1 \land b d > 0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to -\frac{2\,d}{f\,\sqrt{a+b\,d}}\,\text{Subst}\Big[\int \frac{1}{\sqrt{1-x^2}\,\sqrt{1+\frac{(a-b\,d)\,x^2}{a+b\,d}}}\,dx,\,x,\,\frac{\text{Cos}[e+f\,x]}{1+d\,\text{Sin}[e+f\,x]}\Big]$$

$$\to -\frac{2\,d}{f\,\sqrt{a+b\,d}}\,\text{EllipticF}\Big[\text{ArcSin}\Big[\frac{\text{Cos}[e+f\,x]}{1+d\,\text{Sin}[e+f\,x]}\Big],\,-\frac{a-b\,d}{a+b\,d}\Big]$$

## Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*d/(f*Sqrt[a+b*d])*EllipticF[ArcSin[Cos[e+f*x]/(1+d*Sin[e+f*x])],-(a-b*d)/(a+b*d)] /;
FreeQ[{a,b,d,e,f},x] && LtQ[a^2-b^2,0] && EqQ[d^2,1] && GtQ[b*d,0]
```

**2:** $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2<0\,\land\,b^2>0\,\land\,\neg\,(d^2==1\,\land\,b\,d>0)$

## Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{b\,F[x]}}{\sqrt{d\,F[x]}} == 0$

## Rule: If $a^2-b^2<0\,\land\,b^2>0\,\land\,\neg\,(d^2==1\,\land\,b\,d>0)$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to \frac{\sqrt{\text{Sign}[b]\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{\text{Sign}[b]\,\text{Sin}[e+f\,x]}}\,dx$$

## Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[Sign[b]*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[Sign[b]*Sin[e+f*x]]),x] /;
FreeQ[{a,b,d,e,f},x] && LtQ[a^2-b^2,0] && GtQ[b^2,0] && Not[EqQ[d^2,1] && GtQ[b*d,0]]
```

2. $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0 \land \frac{a+b}{d}>0$

**1:** $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2>0 \land \frac{a+b}{d}>0 \land a^2>0$

## Rule: If $a^2-b^2>0 \land \frac{a+b}{d}>0 \land a^2>0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to -\frac{2\sqrt{a^2}\,\sqrt{-\text{Cot}[e+f\,x]^2}}{a\,f\,\sqrt{a^2-b^2}\,\text{Cot}[e+f\,x]}\sqrt{\frac{a+b}{d}}\,\text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\middle/\sqrt{\frac{a+b}{d}}\right],-\frac{a+b}{a-b}\right]$$

## Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*Sqrt[a^2]*Sqrt[-Cot[e+f*x]^2]/(a*f*Sqrt[a^2-b^2]*Cot[e+f*x])*Rt[(a+b)/d,2]*
    EllipticF[ArcSin[Sqrt[a+b*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]/Rt[(a+b)/d,2]],-(a+b)/(a-b)] /;
FreeQ[{a,b,d,e,f},x] && GtQ[a^2-b^2,0] && PosQ[(a+b)/d] && GtQ[a^2,0]
```

**2:** $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0 \land \frac{a+b}{d}>0$

- Rule: If $a^2-b^2\neq 0 \land \frac{a+b}{d}>0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to$$

$$-\frac{2\,\text{Tan}[e+f\,x]}{a\,f}\sqrt{\frac{a+b}{d}}\sqrt{\frac{a\,(1-\text{Csc}[e+f\,x])}{a+b}}\sqrt{\frac{a\,(1+\text{Csc}[e+f\,x])}{a-b}}\,\text{EllipticF}\left[\text{ArcSin}\left[\frac{\sqrt{a+b\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\Big/\sqrt{\frac{a+b}{d}}\right],-\frac{a+b}{a-b}\right]$$

- Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  -2*Tan[e+f*x]/(a*f)*Rt[(a+b)/d,2]*Sqrt[a*(1-Csc[e+f*x])/(a+b)]*Sqrt[a*(1+Csc[e+f*x])/(a-b)]*
    EllipticF[ArcSin[Sqrt[a+b*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]/Rt[(a+b)/d,2]],-(a+b)/(a-b)] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && PosQ[(a+b)/d]
```

3: $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{d\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2\neq 0\ \land\ \frac{a+b}{d}\ngtr 0$

- Derivation: Piecewise constant extraction

- Basis: $\partial_x \frac{\sqrt{-F[x]}}{\sqrt{F[x]}} == 0$

- Rule: If $a^2-b^2\neq 0\ \land\ \frac{a+b}{d}\ngtr 0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{d\,\text{Sin}[e+f\,x]}}\,dx \to \frac{\sqrt{-d\,\text{Sin}[e+f\,x]}}{\sqrt{d\,\text{Sin}[e+f\,x]}}\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{-d\,\text{Sin}[e+f\,x]}}\,dx$$

- Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[-d*Sin[e+f*x]]/Sqrt[d*Sin[e+f*x]]*Int[1/(Sqrt[a+b*Sin[e+f*x]]*Sqrt[-d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0] && NegQ[(a+b)/d]
```

2. $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\ \land\ a^2-b^2\neq 0\ \land\ c^2-d^2\neq 0$

1: $\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\ \land\ a^2-b^2\neq 0\ \land\ c^2-d^2\neq 0\ \land\ \frac{c+d}{a+b}>0$

- Note: Alternative antiderivative contributed via email by Martin Welz on 12 April 2014.

- Rule: If $b\,c-a\,d\neq 0\ \land\ a^2-b^2\neq 0\ \land\ c^2-d^2\neq 0\ \land\ \frac{c+d}{a+b}>0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\ \sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \to$$

$$\frac{2\,(c+d\,\mathrm{Sin}[e+f\,x])}{f\,(b\,c-a\,d)\,\sqrt{\frac{c+d}{a+b}}\,\mathrm{Cos}[e+f\,x]}\sqrt{\frac{(b\,c-a\,d)\,(1-\mathrm{Sin}[e+f\,x])}{(a+b)\,(c+d\,\mathrm{Sin}[e+f\,x])}}$$

$$\sqrt{-\frac{(b\,c-a\,d)\,(1+\mathrm{Sin}[e+f\,x])}{(a-b)\,(c+d\,\mathrm{Sin}[e+f\,x])}}\;\mathrm{EllipticF}\left[\mathrm{ArcSin}\left[\sqrt{\frac{c+d}{a+b}}\,\frac{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}}{\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\right],\,\frac{(a+b)\,(c-d)}{(a-b)\,(c+d)}\right]$$

$$\int\frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx\;\rightarrow$$

$$\frac{2\,(1-\mathrm{Sin}[e+f\,x])}{f\,\sqrt{-\frac{a+b}{a-b}}\,\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\sqrt{\frac{a+b\,\mathrm{Sin}[e+f\,x]}{(a-b)\,(1-\mathrm{Sin}[e+f\,x])}}$$

$$\sqrt{\frac{c+d\,\mathrm{Sin}[e+f\,x]}{(c-d)\,(1-\mathrm{Sin}[e+f\,x])}}\;\mathrm{EllipticF}\left[\mathrm{ArcSin}\left[\sqrt{-\frac{a+b}{a-b}}\,\frac{1+\mathrm{Sin}[e+f\,x]}{\mathrm{Cos}[e+f\,x]}\right],\,\frac{(a-b)\,(c+d)}{(a+b)\,(c-d)}\right]$$

## Program code:

```
Int[1/(Sqrt[a_+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  2*(c+d*Sin[e+f*x])/(f*(b*c-a*d)*Rt[(c+d)/(a+b),2]*Cos[e+f*x])*
    Sqrt[(b*c-a*d)*(1-Sin[e+f*x])/((a+b)*(c+d*Sin[e+f*x]))]*
    Sqrt[-(b*c-a*d)*(1+Sin[e+f*x])/((a-b)*(c+d*Sin[e+f*x]))]*
    EllipticF[ArcSin[Rt[(c+d)/(a+b),2]*(Sqrt[a+b*Sin[e+f*x]]/Sqrt[c+d*Sin[e+f*x]])],(a+b)*(c-d)/((a-b)*(c+d))] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && PosQ[(c+d)/(a+b)]
```

**2:** $\int\frac{1}{\sqrt{a+b\,\mathrm{Sin}[e+f\,x]}\,\sqrt{c+d\,\mathrm{Sin}[e+f\,x]}}\,dx$ when $b\,c-a\,d\neq 0\,\land\,a^2-b^2\neq 0\,\land\,c^2-d^2\neq 0\,\land\,\frac{c+d}{a+b}\ngtr 0$

## Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{\sqrt{-F[x]}}{\sqrt{F[x]}} == 0$

- Rule: If $b\,c-a\,d \neq 0 \land a^2-b^2 \neq 0 \land c^2-d^2 \neq 0 \land \frac{c+d}{a+b} \not> 0$, then

$$\int \frac{1}{\sqrt{a+b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx \to \frac{\sqrt{-a-b\,\text{Sin}[e+f\,x]}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\int \frac{1}{\sqrt{-a-b\,\text{Sin}[e+f\,x]}\,\sqrt{c+d\,\text{Sin}[e+f\,x]}}\,dx$$

- Program code:

```
Int[1/(Sqrt[a_.+b_.*sin[e_.+f_.*x_]]*Sqrt[c_+d_.*sin[e_.+f_.*x_]]),x_Symbol] :=
  Sqrt[-a-b*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]]*Int[1/(Sqrt[-a-b*Sin[e+f*x]]*Sqrt[c+d*Sin[e+f*x]]),x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && NegQ[(c+d)/(a+b)]
```

**9:** $\int \frac{(d\,\text{Sin}[e+f\,x])^{3/2}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$ when $a^2-b^2 \neq 0$

Derivation: Algebraic expansion

Basis: $(d\,z)^{3/2} == -\frac{a\,d\,\sqrt{d\,z}}{2\,b} + \frac{d\,\sqrt{d\,z}\,(a+2\,b\,z)}{2\,b}$

Rule: If $a^2-b^2 \neq 0$, then

$$\int \frac{(d\,\text{Sin}[e+f\,x])^{3/2}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx \to -\frac{a\,d}{2\,b}\int \frac{\sqrt{d\,\text{Sin}[e+f\,x]}}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx + \frac{d}{2\,b}\int \frac{\sqrt{d\,\text{Sin}[e+f\,x]}\,(a+2\,b\,\text{Sin}[e+f\,x])}{\sqrt{a+b\,\text{Sin}[e+f\,x]}}\,dx$$

Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^(3/2)/Sqrt[a_.+b_.*sin[e_.+f_.*x_]],x_Symbol] :=
  -a*d/(2*b)*Int[Sqrt[d*Sin[e+f*x]]/Sqrt[a+b*Sin[e+f*x]],x] +
  d/(2*b)*Int[Sqrt[d*Sin[e+f*x]]*(a+2*b*Sin[e+f*x])/Sqrt[a+b*Sin[e+f*x]],x] /;
FreeQ[{a,b,d,e,f},x] && NeQ[a^2-b^2,0]
```

**10:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,0<m<2\,\wedge\,-1<n<2$

Derivation: Nondegenerate sine recurrence 1b with $A\to a\,c$, $B\to b\,c+a\,d$, $C\to b\,d$, $m\to m-1$, $n\to n-1$, $p\to 0$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0\,\wedge\,0<m<2\,\wedge\,-1<n<2$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\longrightarrow$$
$$-\frac{b\,\mathrm{Cos}[e+f\,x]\,(a+b\,\mathrm{Sin}[e+f\,x])^{m-1}\,(c+d\,\mathrm{Sin}[e+f\,x])^n}{f\,(m+n)}+$$
$$\frac{1}{d\,(m+n)}\int (a+b\,\mathrm{Sin}[e+f\,x])^{m-2}\,(c+d\,\mathrm{Sin}[e+f\,x])^{n-1}\cdot$$
$$\left(a^2\,c\,d\,(m+n)+b\,d\,(b\,c\,(m-1)+a\,d\,n)+(a\,d\,(2\,b\,c+a\,d)\,(m+n)-b\,d\,(a\,c-b\,d\,(m+n-1)))\,\mathrm{Sin}[e+f\,x]+b\,d\,(b\,c\,n+a\,d\,(2\,m+n-1))\,\mathrm{Sin}[e+f\,x]^2\right)dx$$

Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  -b*Cos[e+f*x]*(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n/(f*(m+n)) +
  1/(d*(m+n))*Int[(a+b*Sin[e+f*x])^(m-2)*(c+d*Sin[e+f*x])^(n-1)*
    Simp[a^2*c*d*(m+n)+b*d*(b*c*(m-1)+a*d*n)+
      (a*d*(2*b*c+a*d)*(m+n)-b*d*(a*c-b*d*(m+n-1)))*Sin[e+f*x]+
      b*d*(b*c*n+a*d*(2*m+n-1))*Sin[e+f*x]^2,x],x] /;
FreeQ[{a,b,c,d,e,f},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0] && LtQ[0,m,2] && LtQ[-1,n,2] && NeQ[m+n,0] &&
  (IntegerQ[m] || IntegersQ[2*m,2*n])
```

**11:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,m\in\mathbb{Z}^+$

Derivation: Algebraic expansion

Basis: $a+b\,z==\frac{b\,(c+d\,z)}{d}-\frac{b\,c-a\,d}{d}$

Rule: If $b\,c-a\,d\neq 0\,\wedge\,m\in\mathbb{Z}^+$, then

$$\int (a+b\,\text{Sin}[e+f\,x])^m\,(c+d\,\text{Sin}[e+f\,x])^n\,dx \longrightarrow$$
$$\frac{b}{d}\int (a+b\,\text{Sin}[e+f\,x])^{m-1}\,(c+d\,\text{Sin}[e+f\,x])^{n+1}\,dx-\frac{b\,c-a\,d}{d}\int (a+b\,\text{Sin}[e+f\,x])^{m-1}\,(c+d\,\text{Sin}[e+f\,x])^n\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  b/d*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^(n+1),x] -
  (b*c-a*d)/d*Int[(a+b*Sin[e+f*x])^(m-1)*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && NeQ[b*c-a*d,0] && IGtQ[m,0]
```

**12.** $\int (d\,\text{Sin}[e+f\,x])^n\,(a+b\,\text{Sin}[e+f\,x])^m\,dx$ when $a^2-b^2=0\,\land\,m\in\mathbb{Z}^-$

**1:** $\int \frac{(d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx$ when $a^2-b^2\neq 0$

## Derivation: Algebraic expansion

– Basis: $\frac{1}{a+b\,z}==\frac{a}{a^2-b^2\,z^2}-\frac{b\,z}{a^2-b^2\,z^2}$

– Rule: If $a^2-b^2\neq 0$, then

$$\int \frac{(d\,\text{Sin}[e+f\,x])^n}{a+b\,\text{Sin}[e+f\,x]}\,dx \longrightarrow a\int \frac{(d\,\text{Sin}[e+f\,x])^n}{a^2-b^2\,\text{Sin}[e+f\,x]^2}\,dx-\frac{b}{d}\int \frac{(d\,\text{Sin}[e+f\,x])^{n+1}}{a^2-b^2\,\text{Sin}[e+f\,x]^2}\,dx$$

– 

## Program code:

```
Int[(d_.*sin[e_.+f_.*x_])^n_./(a_+b_.*sin[e_.+f_.*x_]),x_Symbol] :=
  a*Int[(d*Sin[e+f*x])^n/(a^2-b^2*Sin[e+f*x]^2),x] -
  b/d*Int[(d*Sin[e+f*x])^(n+1)/(a^2-b^2*Sin[e+f*x]^2),x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0]
```

**2:** $\int (d\,\mathrm{Sin}[e+f\,x])^n\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,dx$ when $a^2-b^2\neq 0\,\wedge\,m+1\in\mathbb{Z}^-$

## Derivation: Algebraic expansion

## Basis: $\frac{1}{a+b\,z}==\frac{a-b\,z}{a^2-b^2\,z^2}$

## Rule: If $a^2-b^2\neq 0\,\wedge\,m+1\in\mathbb{Z}^-$, then

$$\int (d\,\mathrm{Sin}[e+f\,x])^n\,(a+b\,\mathrm{Sin}[e+f\,x])^m\,dx\,\rightarrow\,\int \mathrm{ExpandTrig}\Big[\frac{(d\,\mathrm{Sin}[e+f\,x])^n\,(a-b\,\mathrm{Sin}[e+f\,x])^{-m}}{\left(a^2-b^2\,\mathrm{Sin}[e+f\,x]^2\right)^{-m}},\,x\Big]\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(d_.*sin[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[ExpandTrig[(d*sin[e+f*x])^n*(a-b*sin[e+f*x])^(-m)/(a^2-b^2*sin[e+f*x]^2)^(-m),x],x] /;
FreeQ[{a,b,d,e,f,n},x] && NeQ[a^2-b^2,0] && ILtQ[m,-1]
```

**X:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$ when $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$

## Rule: If $b\,c-a\,d\neq 0\,\wedge\,a^2-b^2\neq 0\,\wedge\,c^2-d^2\neq 0$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx\,\rightarrow\,\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Sin}[e+f\,x])^n\,dx$$

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_*(c_.+d_.*sin[e_.+f_.*x_])^n_,x_Symbol] :=
  Unintegrable[(a+b*Sin[e+f*x])^m*(c+d*Sin[e+f*x])^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && NeQ[b*c-a*d,0] && NeQ[a^2-b^2,0] && NeQ[c^2-d^2,0]
```

## Rules for integrands of the form $(a + b\,\text{Sin}[e + f\,x])^m\,(c\,(d\,\text{Sin}[e + f\,x])^p)^n$

**x:** $\int (a + b\,\text{Sin}[e + f\,x])^m\,(d\,\text{Csc}[e + f\,x])^n\,dx$ when $n \notin \mathbb{Z} \land m \in \mathbb{Z}$

Derivation: Algebraic normalization

Basis: If $m \in \mathbb{Z}$, then $(a + b\,\text{Sin}[z])^m == \frac{d^m\,(b+a\,\text{Csc}[z])^m}{(d\,\text{Csc}[z])^m}$

Note: Although this rule does not introduce a piecewise constant factor, it is better to stay in the sine/cosine world than the secant/cosecant world.

Rule: If $n \notin \mathbb{Z} \land m \in \mathbb{Z}$, then

$$\int (a + b\,\text{Sin}[e + f\,x])^m\,(d\,\text{Csc}[e + f\,x])^n\,dx \longrightarrow d^m \int (d\,\text{Csc}[e + f\,x])^{n-m}\,(b + a\,\text{Csc}[e + f\,x])^m\,dx$$

Program code:

```
(* Int[(a_.+b_.*sin[e_.+f_.*x_])^m_.*(d_./sin[e_.+f_.*x_])^n_,x_Symbol] :=
  d^m*Int[(d*Csc[e+f*x])^(n-m)*(b+a*Csc[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m] *)
```

```
(* Int[(a_.+b_.*cos[e_.+f_.*x_])^m_.*(d_./cos[e_.+f_.*x_])^n_,x_Symbol] :=
  d^m*Int[(d*Sec[e+f*x])^(n-m)*(b+a*Sec[e+f*x])^m,x] /;
FreeQ[{a,b,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m] *)
```

**1:** $\int (a + b \operatorname{Sin}[e + f x])^m \left(c \left(d \operatorname{Sin}[e + f x]\right)^p\right)^n \, dx$ when $n \notin \mathbb{Z}$

## Derivation: Piecewise constant extraction

## Basis: $\partial_x \frac{\left(c \left(d \operatorname{Sin}[e+f x]\right)^p\right)^n}{\left(d \operatorname{Sin}[e+f x]\right)^{n p}} == 0$

–

## Rule: If $n \notin \mathbb{Z}$, then

$$\int (a + b \operatorname{Sin}[e + f x])^m \left(c \left(d \operatorname{Sin}[e + f x]\right)^p\right)^n \, dx \longrightarrow \frac{c^{\operatorname{IntPart}[n]} \left(c \left(d \operatorname{Sin}[e + f x]\right)^p\right)^{\operatorname{FracPart}[n]}}{\left(d \operatorname{Sin}[e + f x]\right)^{p \operatorname{FracPart}[n]}} \int (a + b \operatorname{Sin}[e + f x])^m \left(d \operatorname{Sin}[e + f x]\right)^{n p} \, dx$$

–

## Program code:

```
Int[(a_.+b_.*sin[e_.+f_.*x_])^m_.*(c_.*(d_.*sin[e_.+f_.*x_])^p_)^n_,x_Symbol] :=
  c^IntPart[n]*(c*(d*Sin[e + f*x])^p)^FracPart[n]/(d*Sin[e + f*x])^(p*FracPart[n])*
    Int[(a+b*Sin[e+f*x])^m*(d*Sin[e+f*x])^(n*p),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[n]]
```

```
Int[(a_.+b_.*cos[e_.+f_.*x_])^m_.*(c_.*(d_.*cos[e_.+f_.*x_])^p_)^n_,x_Symbol] :=
  c^IntPart[n]*(c*(d*Cos[e + f*x])^p)^FracPart[n]/(d*Cos[e + f*x])^(p*FracPart[n])*
    Int[(a+b*Cos[e+f*x])^m*(d*Cos[e+f*x])^(n*p),x] /;
FreeQ[{a,b,c,d,e,f,m,n,p},x] && Not[IntegerQ[n]]
```

# Rules for integrands of the form $(a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n$

**1:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx$ when $n\in\mathbb{Z}$

Derivation: Algebraic normalization

Basis: $c+d\,\mathrm{Csc}[z]==\frac{d+c\,\mathrm{Sin}[z]}{\mathrm{Sin}[z]}$

Rule: If $n\in\mathbb{Z}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx\;\longrightarrow\;\int\frac{(a+b\,\mathrm{Sin}[e+f\,x])^m\,(d+c\,\mathrm{Sin}[e+f\,x])^n}{\mathrm{Sin}[e+f\,x]^n}\,dx$$

Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_.,x_Symbol] :=
  Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m},x] && IntegerQ[n]
```

**2.** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx$ when $n\notin\mathbb{Z}$

**1:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx$ when $n\notin\mathbb{Z}\,\wedge\,m\in\mathbb{Z}$

Derivation: Algebraic normalization

Basis: $a+b\,\mathrm{Sin}[z]==\frac{b+a\,\mathrm{Csc}[z]}{\mathrm{Csc}[z]}$

Rule: If $n\notin\mathbb{Z}\,\wedge\,m\in\mathbb{Z}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx \to \int \frac{(b+a\,\mathrm{Csc}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n}{\mathrm{Csc}[e+f\,x]^m}\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_.*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(b+a*Csc[e+f*x])^m*(c+d*Csc[e+f*x])^n/Csc[e+f*x]^m,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m]
```

```
Int[(a_+b_.*cos[e_.+f_.*x_])^m_.*(c_+d_.*sec[e_.+f_.*x_])^n_,x_Symbol] :=
  Int[(b+a*Sec[e+f*x])^m*(c+d*Sec[e+f*x])^n/Sec[e+f*x]^m,x] /;
FreeQ[{a,b,c,d,e,f,n},x] && Not[IntegerQ[n]] && IntegerQ[m]
```

**2:** $\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx$ when $n \notin \mathbb{Z} \land m \notin \mathbb{Z}$

## Derivation: Piecewise constant extraction

Basis: $\partial_x \frac{(c+d\,\mathrm{Csc}[e+f\,x])^n\,\mathrm{Sin}[e+f\,x]^n}{(d+c\,\mathrm{Sin}[e+f\,x])^n} == 0$

Rule: If $n \notin \mathbb{Z} \land m \notin \mathbb{Z}$, then

$$\int (a+b\,\mathrm{Sin}[e+f\,x])^m\,(c+d\,\mathrm{Csc}[e+f\,x])^n\,dx \to \frac{(c+d\,\mathrm{Csc}[e+f\,x])^n\,\mathrm{Sin}[e+f\,x]^n}{(d+c\,\mathrm{Sin}[e+f\,x])^n}\int \frac{(a+b\,\mathrm{Sin}[e+f\,x])^m\,(d+c\,\mathrm{Sin}[e+f\,x])^n}{\mathrm{Sin}[e+f\,x]^n}\,dx$$

## Program code:

```
Int[(a_+b_.*sin[e_.+f_.*x_])^m_*(c_+d_.*csc[e_.+f_.*x_])^n_,x_Symbol] :=
  Sin[e+f*x]^n*(c+d*Csc[e+f*x])^n/(d+c*Sin[e+f*x])^n*Int[(a+b*Sin[e+f*x])^m*(d+c*Sin[e+f*x])^n/Sin[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[n]] && Not[IntegerQ[m]]
```

```
Int[(a_+b_.*cos[e_.+f_.*x_])^m_*(c_+d_.*sec[e_.+f_.*x_])^n_,x_Symbol] :=
  Cos[e+f*x]^n*(c+d*Sec[e+f*x])^n/(d+c*Cos[e+f*x])^n*Int[(a+b*Cos[e+f*x])^m*(d+c*Cos[e+f*x])^n/Cos[e+f*x]^n,x] /;
FreeQ[{a,b,c,d,e,f,m,n},x] && Not[IntegerQ[n]] && Not[IntegerQ[m]]
```